Piterbarg Theorems for Chi-processes with Trend

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Abstract: Let $\chi_n(t) = (\sum_{i=1}^n X_i^2(t))^{1/2},\ t \ge 0$ be a chi-process with n degrees of freedom where X_i's are independent copies of some generic centred Gaussian process X. This paper derives the exact asymptotic behaviour of

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\Big(\chi_n(t)-g(t)\Big)>u\right\} \quad \text{as} \quad u\to\infty,$$

where T is a given positive constant, and $g(\cdot)$ is some non-negative bounded measurable function. The case $g(t)\equiv 0$ has been investigated in numerous contributions by V.I. Piterbarg. Our novel asymptotic results, for both stationary and non-stationary X, are referred to as Piterbarg theorems for chi-processes with trend.

Key words: Gaussian random fields; Piterbarg theorem for chi-process; Pickands constant; generalized Piterbarg constant; Piterbarg inequality.
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1 Introduction

Two fundamental results for the study of asymptotic behaviour of the supremum of non-smooth Gaussian processes and Gaussian random fields are *Pickands theorem* and *Piterbarg theorem*, see Pickands (1969a,b), Piterbarg (1972, 1996), and Piterbarg and Prisyazhnyuk (1978). For any fixed $T\in(0,\infty)$, J. Pickands III obtained the exact tail asymptotics of $\sup_{t\in[0,T]} X(t)$ for a centered stationary Gaussian process $\{X(t), t\ge 0\}$ with a.s. continuous sample paths and covariance function $r(\cdot)$ satisfying the following assumptions:
Assumption I: $r(t) = 1-|t|^\alpha(1+o(1))$ as $t\to 0$, with $\alpha\in(0,2]$;
Assumption II: $r(t)<1$ for all $t>0$.
More precisely, *Pickands theorem* states that

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t)>u\right\} = \mathcal{H}_\alpha T\frac{1}{\sqrt{2\pi}}u^{\frac{2}{\alpha}-1}\exp\left(-\frac{u^2}{2}\right)(1+o(1)) \quad \text{as} \quad u\to\infty, \tag{1.1}$$

where $\mathcal{H}_\alpha$ is the *Pickands constant* defined by

$$\mathcal{H}_\alpha = \lim_{S\to\infty}\frac{1}{S}\mathbb{E}\left\{\exp\Big(\sup_{t\in[0,S]}\Big(\sqrt{2}B_\alpha(t)-t^\alpha\Big)\Big)\right\}\in(0,\infty),$$

with $\{B_\alpha(t), t\in\mathbb{R}\}$ a standard fractional Brownian motion (fBm) defined on $\mathbb{R}$ with Hurst index $\alpha/2\in(0,1]$. J. Pickands III proved (1.1) using the *double sum method* and the following asymptotics (set $S\in(0,\infty)$)

$$\mathbb{P}\left\{\sup_{t\in[0,u^{-2/\alpha}S]} X(t)>u\right\} = \mathcal{H}_\alpha[0,S]\frac{1}{\sqrt{2\pi}u}\exp\left(-\frac{u^2}{2}\right)(1+o(1)) \quad \text{as} \quad u\to\infty, \tag{1.2}$$

where

$$\mathcal{H}_\alpha[0,S] = \mathbb{E}\left\{\exp\Big(\sup_{t\in[0,S]}\Big(\sqrt{2}B_\alpha(t)-t^\alpha\Big)\Big)\right\}\in(0,\infty).$$

Piterbarg (1972, 1996) obtained a similar result for non-stationary Gaussian processes, namely

$$\mathbb{P}\left\{\sup_{t\in[0,u^{-2/\alpha}S]}\frac{X(t)}{1+dt^\alpha}>u\right\} = \mathcal{P}_{\alpha,\alpha}^d[0,S]\frac{1}{\sqrt{2\pi}u}\exp\left(-\frac{u^2}{2}\right)(1+o(1)) \quad \text{as} \quad u\to\infty, \tag{1.3}$$

where X is the centered stationary Gaussian process as above, $d>0$, and

$$\mathcal{P}_{\alpha,\beta}^{d}[0,S]=\mathbb{E}\left\{\exp\left(\sup_{t\in[0,S]}\left(\sqrt{2}B_\alpha(t)-|t|^\alpha-d|t|^\beta\right)\right)\right\}\in(0,\infty),\quad d,\beta,S\in(0,\infty).$$

For a centered non-stationary Gaussian process $\{X(t),t\ge 0\}$ with a.s. continuous sample paths the next two assumptions are crucial:

Assumption III: The standard deviation function $\sigma_X(\cdot)$ of X attains its maximum (assumed to be 1) over $[0,T]$ at the unique point $t=T$. Further, there exist some positive constants $\nu\in(0,2],\mu,A,D$ such that

$$\sigma_X(t) \quad = \quad 1-A(T-t)^\mu+o((T-t)^\mu),\quad t\uparrow T, \tag{1.4}$$

and

$$r_X(s,t)=Corr(X(s),X(t))=1-D|t-s|^\nu+o(|t-s|^\nu),\quad s,t\uparrow T. \tag{1.5}$$

Assumption IV: There exist positive constants G and γ such that

$$\mathbb{E}\left\{(X(t)-X(s))^2\right\} \quad \le \quad G|t-s|^\gamma \tag{1.6}$$

holds for all $s,t\in[0,T]$.

For such a centered non-stationary Gaussian process $\{X(t),t\ge 0\}$ it is known that (see e.g., Dębicki and Sikora (2011), Theorem D.3 in Piterbarg (1996) or Theorem 2.1 in Dębicki et al. (2014))

$$\mathbb{P}\left\{\sup_{t\in[0,T]}X(t)>u\right\}=\mathcal{D}_{\nu,\mu}\frac{1}{\sqrt{2\pi}}u^{\left(\frac{2}{\nu}-\frac{2}{\mu}\right)_+-1}\exp\left(-\frac{u^2}{2}\right)(1+o(1))\quad\text{as}\quad u\to\infty,$$

where $(x)_+=\max(0,x)$, and $\mathcal{D}_{\nu,\mu}$ is a positive constant, which, when $\mu=\nu$ is commonly referred to as the *Piterbarg constant* defined by

$$\mathcal{P}_{\nu,\nu}^{\frac{A}{D}}=\lim_{S\to\infty}\mathcal{P}_{\nu,\nu}^{\frac{A}{D}}[0,S]\in(0,\infty).$$

It is worth pointing out that in Theorem D.3 in Piterbarg (1996) it is assumed that the unique maximum point of $\sigma_X(\cdot)$ is attained at some inner point of $(0,T)$; in that case the Piterbarg constant is given by $\tilde{\mathcal{P}}_{\nu,\nu}^{\frac{A}{D}}=\lim_{S\to\infty}\mathcal{P}_{\nu,\nu}^{\frac{A}{D}}[-S,S]$.

Let $\{\chi_n(t),t\ge 0\}$ be a chi-process with $n\in\mathbb{N}$ degrees of freedom defined by

$$\chi_n(t) \quad = \quad \sqrt{\sum_{i=1}^n X_i^2(t)},\quad t\ge 0,$$

where $\{X_i(t),t\ge 0\}$, $1\le i\le n$, are independent copies of a centered Gaussian process $\{X(t),t\ge 0\}$ with a.s. continuous sample paths. The investigation of

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\chi_n(t)>u\right\}\quad\text{as } u\to\infty \tag{1.7}$$

was initiated by an envelope of a Gaussian process over a high level, see e.g., Belyaev and Nosko (1969), Lindgren (1980a,b, 1989). The tail asymptotic behaviour of chi-processes is crucial for numerous statistical applications, see e.g., Aronowich and Adler (1985), Albin and Jarušková (2003), Jarušková (2010), Jarušková and Piterbarg (2011), and the references therein. We mention in passing that the limit behaviour of maximum of chi-processes is the same as that for Gaussian processes (Kabluchko (2011), Hashorva et al. (2012)); in the limit the Brown-Resnick process appears.

Albin (1990) studied the exact asymptotics of (1.7) for a centered stationary generalized chi-process using Berman's approach (see Berman (1992) and Albin (1998) for self-similar chi-processes), whereas Piterbarg (1994a) obtained a generalization of Albin's result by resorting to *the double sum method*. In Piterbarg (1994b), the author investigated the exact asymptotics of (1.7) for a centered non-stationary generalized chi-process where the generic Gaussian process is differentiable and with variance attaining its global maximum at only one inner point of the interval $[0,T]$. Throughout the paper, a chi-process generated by centered (non-)stationary Gaussian processes is called a *(non-)stationary chi-process*.

Let $g(\cdot)$ be a non-negative bounded measurable function satisfying one of the following two conditions:
Assumption V: $g(\cdot)$ attains its minimum 0 over $[0,T]$ at the unique point 0, and further there exist some positive constants c,β such that

$$g(t)=ct^{\beta}(1+o(1)),\quad t\to 0;$$

Assumption VI: There exist some constants $\widetilde{c}\in\mathbb{R}$ and $\widetilde{\beta}>0$ such that

$$g(t)=g(T)-\widetilde{c}(T-t)^{\widetilde{\beta}}(1+o(1)),\quad t\to T.$$

In this paper, we derive the exact asymptotics of

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\Big(\chi_n(t)-g(t)\Big)>u\right\}\quad \text{as } u\to\infty \tag{1.8}$$

for i) stationary chi-processes with a trend function $g(\cdot)$ satisfying Assumption V; ii) non-stationary chi-processes with a trend function $g(\cdot)$ satisfying Assumption VI.
The investigation of the tail asymptotics of the maximum of chi-processes with trend is motivated by the problem of the exit of a vector Gaussian load process in engineering sciences, see, e.g., Lindgren (1980a) and the references therein. More precisely, let $\boldsymbol{X}(t)=(X_1(t),\cdots,X_n(t)),t\ge 0$ be a vector Gaussian load process. Of interest is the probability of exit

$$\mathbb{P}\left\{\boldsymbol{X}(t)\notin \boldsymbol{S}_u(t),\quad \text{for some } t\in[0,T]\right\},$$

with a *time-dependent safety region*

$$\boldsymbol{S}_u(t)=\left\{(x_1,\cdots,x_n)\in\mathbb{R}^n:\sqrt{\sum_{i=1}^n x_i^2}\le h(t,u)\right\}.$$

The model where $h(t,u)\equiv u$ was considered extensively in the literature as mentioned above; the model where $h(t,u)=u\times d(t)$, with $d(\cdot)$ a positive measurable function, was mentioned in Kozachenko and Moklyachuk (1999) where the authors mainly focused on the exit problem of a class of square-Gaussian processes. In this paper we shall consider a tractable case that $h(t,u)=u+g(t)$, with $g(\cdot)$ defined as above. The obtained results might also be useful in reliability theory and mathematical statistics applications. The analysis of (1.8) is based on a tailored *double sum method* for chi-processes. Surprisingly, a *generalized Piterbarg constant* $\mathcal{P}^d_{\alpha,\beta}$, with $\alpha\in(0,2],\beta=\alpha/2,d>0$, defined by

$$\mathcal{P}^d_{\alpha,\beta}=\lim_{S\to\infty}\mathcal{P}^d_{\alpha,\beta}[0,S]\in(0,\infty)$$

appears in the asymptotics of the stationary chi-process with trend (we do not observe a generalized Pickands constant as in Dębicki (2002)).

Organization of the paper: The main results for the stationary and non-stationary chi-processes with trend are given in Section 2. The proofs are relegated to Section 3 which is followed then by an Appendix.

2 Main Results

In order to avoid repetitions we shall consider below a chi-process $\{\chi_n(t), t \ge 0\}$ as defined above by taking independent copies of a generic centered Gaussian processes X with a.s. continuous sample paths. Our asymptotic results will thus depend on the properties of the Gaussian process X. As expected, the stationary case is completely different compared with the non-stationary one. Throughout this paper denote

$$\Upsilon_n(u) := \frac{2^{(2-n)/2}}{\Gamma(n/2)} u^{n-2} \exp\left(-\frac{u^2}{2}\right),$$

which is the asymptotic expansion of the survival function of $\chi_n(0)$ i.e.,

$$\mathbb{P}\left\{\chi_n(0) > u\right\} = \Upsilon_n(u)(1+o(1)) \quad \text{as } u \to \infty,$$

provided that $X(0)$ is standard normal (i.e., an $N(0,1)$ random variable).
We first present two preliminary results on the tail asymptotics of the maximum of stationary chi-processes without trend. The next result can be found in Corollary 7.3 in Piterbarg (1996).

Proposition 2.1 *Let $\{X(t), t \ge 0\}$ be a stationary Gaussian process with covariance function $r(\cdot)$ satisfying Assumption I and Assumption II with $\alpha \in (0,2]$. Then, for any constant $T \in (0,\infty)$*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \chi_n(t) > u\right\} = T\mathcal{H}_\alpha u^{\frac{2}{\alpha}} \Upsilon_n(u)(1+o(1)) \tag{2.9}$$

holds as $u \to \infty$.

An implication of is the following proposition which will play an important role in the proof of our main results; it can be derived by examining the arguments in Piterbarg (1996).

Proposition 2.2 *Let $f(\cdot)$ be a positive function defined in $[0,\infty)$ such that $\lim_{u\to\infty} f(u)/u = 1$ and let $S \in (0,\infty)$ be a constant. Under the assumptions of Proposition 2.1 we have that*

$$\mathbb{P}\left\{\sup_{t\in[0,u^{-2/\alpha}S]} \chi_n(t) > f(u)\right\} = \mathcal{H}_\alpha[0,S]\Upsilon_n(f(u))(1+o(1)) \tag{2.10}$$

holds as $u \to \infty$.

It is worth mentioning that Propositions 2.1 and 2.2 are parallel results of Pickands for chi-processes; see (1.1) and (1.2).
Next, we give our first result concerning the exact tail asymptotics of the supremum of stationary chi-processes with trend.

Theorem 2.3 *Suppose that the covariance function $r(\cdot)$ of the centered stationary Gaussian process $\{X(t), t \ge 0\}$ satisfies Assumption I and Assumption II with $\alpha \in (0,2]$. Assume further that $g(\cdot)$ satisfies Assumption V with the parameters therein. Then*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \Big(\chi_n(t) - g(t)\Big) > u\right\} = \mathcal{M}^c_{\alpha,\beta} u^{(\frac{2}{\alpha}-\frac{1}{\beta})_+} \Upsilon_n(u)(1+o(1)) \tag{2.11}$$

as $u \to \infty$, where

$$\mathcal{M}^c_{\alpha,\beta} = \begin{cases} c^{-1/\beta}\Gamma(1/\beta+1)\mathcal{H}_\alpha, & \text{if } \alpha < 2\beta, \\ \mathcal{P}^c_{\alpha,\alpha/2}, & \text{if } \alpha = 2\beta, \\ 1 & \text{if } \alpha > 2\beta. \end{cases}$$

Remarks 2.4 *a) For any $d>0$*

$$\mathcal{P}_{2,1}^{d}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{d}{\sqrt{2}}}e^{-\frac{x^2}{2}}\,dx+\frac{1}{d\sqrt{\pi}}e^{\frac{d^2}{4}-1}.$$

In general $\mathcal{P}_{\alpha,\alpha/2}^{d}$ is an unknown positive constant which can be eventually calculated by simulations. We mention in passing the paper of Dieker and Yakir (2013) where a new approach is introduced for estimating the Pickands constants.
b) We see from the proof of last theorem that the minimum of the trend function $g(\cdot)$ taking on $[0,T]$ plays a crucial role. If we assume that $t_0=argmin_{t\in[0,T]}g(t)\in(0,T)$ which is unique and further there exist some positive constants c,β such that

$$g(t)=g(t_0)+c|t-t_0|^{\beta}(1+o(1)),\quad t\to t_0,$$

then (2.11) still holds with u replaced by $u+g(t_0)$, $\Gamma(\cdot)$ replaced by $2\Gamma(\cdot)$, and $\mathcal{P}_{\alpha,\alpha/2}^{c}$ replaced by $\tilde{\mathcal{P}}_{\alpha,\alpha/2}^{c}:=\lim_{S\to\infty}\mathcal{P}_{\alpha,\alpha/2}^{c}[-S,S]$.
c) In view of our proofs and the key results of Piterbarg (1994a) it is possible to obtain additional results for generalized chi-processes. For instance, if $\{\chi_n(t),t\ge0\}$ is a generalized stationary chi-process defined by

$$\chi_n(t)\;=\;\sqrt{\sum_{i=1}^{n}b_i^2X_i^2(t)},\quad t\ge0,$$

with $1=b_1=\cdots=b_k>b_{k+1}\ge b_{k+2}\ge\cdots\ge b_n$, for some $1\le k<n$, then under assumptions of Theorem 2.3

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\Big(\chi_n(t)-g(t)\Big)>u\right\}=\prod_{i=k+1}^{n}(1-b_i^2)^{-1/2}\mathcal{M}_{\alpha,\beta}^{c}u^{(\frac{2}{\alpha}-\frac{1}{\beta})_+}\Upsilon_k(u)(1+o(1))$$

as $u\to\infty$. In order to keep a suitable length of the paper and to avoid extra notation we do not consider here general chi-processes.

Examples of X: Numerous important Gaussian processes satisfy the assumptions of Theorem 2.3. We present next two interesting cases:
Fractional Gaussian noise: Consider X to be the fractional Gaussian noise, i.e.,

$$X(t)=B_\alpha(t+1)-B_\alpha(t),\quad t\ge0,$$

with B_α a fBm with Hurst index $\alpha/2\in(0,1)$. For $\alpha=1$, X is also known as Slepian process. Clearly X is stationary for any $\alpha\in(0,2)$ and further the covariance function satisfies

$$r(t)=1-|t|^{\alpha}(1+o(1)),\quad t\to0;\quad \text{and}\ \ r(t)<1 \text{ for all } t>0.$$

Lamperti transformation of fBm: Define the Gaussian process X via Lamperti transform of a fBm, i.e., $X(t)=e^{-\alpha/2t}B_\alpha(e^t)$, which is again a stationary Gaussian process. For the covariance function we have

$$r(t)=1-\frac{1}{2}|t|^{\alpha}(1+o(1)),\quad t\to0;\quad \text{and}\ \ r(t)<1 \text{ for all } t>0.$$

Next, we deal with a large class of non-stationary chi-processes presenting first the result for chi-processes without trend.

Theorem 2.5 *Assume that the centered Gaussian process $\{X(t),t\ge0\}$ satisfies Assumption III and Assumption IV with the constants therein. Then, for any $T_1\in[0,T)$ we have*

$$\mathbb{P}\left\{\sup_{t\in[T_1,T]}\chi_n(t)>u\right\}\;=\;\mathcal{M}_{\nu,\mu}u^{(\frac{2}{\nu}-\frac{2}{\mu})_+}\Upsilon_n(u)(1+o(1))\tag{2.12}$$

as $u \to \infty$, where

$$\mathcal{M}_{\nu,\mu} = \begin{cases} D^{1/\nu} \frac{\Gamma(1/\mu+1)}{A^{1/\mu}} \mathcal{H}_\nu, & \text{if } \nu < \mu, \\ \mathcal{P}_{\nu,\nu}^{\frac{A}{D}}, & \text{if } \nu = \mu, \\ 1 & \text{if } \nu > \mu. \end{cases}$$

We state below an extension of the Piterbarg theorem allowing the non-stationary chi-processes to have a non-zero trend.

Theorem 2.6 *Assume that $g(\cdot)$ is a positive bounded measurable function satisfying Assumption VI. Under the assumptions of Theorem 2.5, if $\mu \le \widetilde{\beta}$, then (set $u_* := u + g(T)$)*

$$\mathbb{P}\left\{ \sup_{t \in [0,T]} \Big(\chi_n(t) - g(t) \Big) > u \right\} = \mathcal{M}_{\nu,\mu} u_*^{\left(\frac{2}{\nu} - \frac{2}{\mu}\right)_+} \Upsilon_n(u_*) (1 + o(1)) \tag{2.13}$$

as $u \to \infty$.

Remarks 2.7 *a) As it can be seen from the last two theorems that the only difference between the cases with and without trend is $g(T)$ in u_*.*
b) We conclude from the proof of Theorem 2.5 that the Assumption IV can be relaxed where it can be assumed that there is some $T_0 \in (T_1, T)$ such that (1.6) holds for all $s, t \in [T_0, T]$.

Examples of X: Several important Gaussian processes satisfy the assumptions of Theorems 2.5 and 2.6. We present below three interesting Gaussian processes (discussed in Houdré and Villa (2003), Bojdecki et al. (2004) and Dębicki and Tabiś (2011), respectively).

Bi-fractional Brownian motion: Consider $B_{K,H}$ with $K, H \in (0,1)$ to be a bi-fBm, i.e., a self-similar Gaussian process with covariance function given by

$$Cov(B_{K,H}(t), B_{K,H}(s)) = \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t-s|^{2KH} \right), \quad t, s \ge 0.$$

It follows that the standard deviation σ of $B_{K,H}$ attaints its maximum over $[0,T]$ at the unique point T and

$$\sigma(t) = T^{KH} - KHT^{KH-1}(T-t)(1+o(1)), \quad t \to T.$$

Further

$$1 - Corr(B_{K,H}(t), B_{K,H}(s)) = \frac{1}{2^K T^{2KH}} |t-s|^{2KH} (1 + o(1)), \quad t, s \to T$$

and for all $s, t \in [0,T]$ there exists some constant $G > 0$ such that

$$\mathbb{E}\left\{ (B_{K,H}(t) - B_{K,H}(s))^2 \right\} \le G|t-s|^{2KH}.$$

Sub-fractional Brownian motion: The sub-fBm S_H with $H \in (0,1)$ is a self-similar Gaussian process with covariance given by

$$Cov(S_H(t), S_H(s)) = t^{2H} + s^{2H} - \frac{1}{2}\left((s+t)^{2H} + |t-s|^{2H} \right), \quad t, s \ge 0.$$

The standard deviation σ of S_H attaints its maximum over $[0,T]$ at the unique point T and

$$\sigma(t) = \sqrt{2 - 2^{2H-1}} T^H - \sqrt{2 - 2^{2H-1}} H T^{H-1} (T-t)(1+o(1)), \quad t \to T.$$

Moreover

$$1 - Corr(S_H(t), S_H(s)) = \frac{1}{2(2 - 2^{2H-1}) T^{2H}} |t-s|^{2H} (1 + o(1)), \quad t, s \to T$$

and, for all $s, t \in [0,T]$, there exists some constant $G > 0$, such that

$$\mathbb{E}\left\{ (S_H(t) - S_H(s))^2 \right\} \le G|t-s|^{H/2}.$$

Mean integrated fBm: Consider a Gaussian process X_H given by

$$X_H(t) = \begin{cases} \sqrt{2H+2}\frac{1}{t}\int_0^t B_{2H}(s)ds, & t>0,\\ 0, & t=0, \end{cases}$$

with $H \in (0,1)$. In view of Dębicki and Tabiś (2011), we conclude that the standard deviation σ of X_H attaints its maximum over $[0,T]$ at the unique point T and

$$\sigma(t) = T^H - HT^{H-1}(T-t)(1+o(1)), \quad t \to T.$$

Further

$$1 - Corr(X_H(t), X_H(s)) = \frac{1}{2T^2}(1-H^2)|t-s|^2(1+o(1)), \quad t,s \to T$$

and, for all $s,t \in [\delta, T]$ with some $\delta \in (0,T)$, there exists some constant $G>0$, such that

$$\mathbb{E}\left\{(X_H(t) - X_H(s))^2\right\} \quad \le \quad G\delta^{-2}|t-s|.$$

3 Further Results and Proofs

In what follows, we give proofs of all the theorems in this paper. Hereafter the positive constant $\mathbb{Q}$ may be different from line to line.

Let $\{\xi_u(t,\boldsymbol{v}), t\ge 0, \boldsymbol{v}\in \mathbb{R}^{n-1}\}$, $u \ge 0$ be a family of centered stationary Gaussian random fields with a.s. continuous sample paths, and covariance function $r_{\xi_u}(t,\boldsymbol{v})$ given by

$$r_{\xi_u}(t,\boldsymbol{v}) = \exp\left(-u^{-2}D_0 t^{\alpha_0} - \sum_{i=1}^{n-1} D_i|v_i|^{\alpha_i}\right), \quad t\ge 0, \boldsymbol{v}\in\mathbb{R}^{n-1}$$

for some positive constants $D_i, 0\le i\le n-1$, and $\alpha_i \in (0,2], 0 \le i \le n-1$.

Theorem 3.1 *Let $f(\cdot)$ be a positive function defined in $[0,\infty)$ such that $\lim_{u\to\infty} f(u)/u = 1$. For any $c,\beta,S_1,S_2>0$ we have*

$$\mathbb{P}\left\{\sup_{\substack{t\in[0,S_1]\\ \boldsymbol{v}\in\prod_{i=1}^{n-1}\left[0,u^{-2/\alpha_i}S_2\right]}} \frac{\xi_u(t,\boldsymbol{v})}{1+ct^\beta u^{-2}} > f(u)\right\} = \mathcal{P}_{\alpha_0,\beta}^{cD_0^{-\frac{\beta}{\alpha_0}}}\left[0, D_0^{\frac{1}{\alpha_0}}S_1\right]\prod_{i=1}^{n-1}\mathcal{H}_{\alpha_i}\left[0, D_i^{\frac{1}{\alpha_i}}S_2\right]$$
$$\times \frac{1}{\sqrt{2\pi}f(u)}\exp\left(-\frac{(f(u))^2}{2}\right)(1+o(1))$$

as $u\to\infty$.

Proof of Theorem 3.1 Set $\zeta_u(t,\boldsymbol{v}) = \xi_u(t, u^{-2/\alpha_1}v_1, \cdots, u^{-2/\alpha_{n-1}}v_{n-1}), t\ge 0, \boldsymbol{v}\in\mathbb{R}^{n-1}, u>0$ with covariance function

$$r_{\zeta_u}(t,\boldsymbol{v}) = \exp\left(-u^{-2}D_0t^{\alpha_0} - u^{-2}\sum_{i=1}^{n-1}D_i|v_i|^{\alpha_i}\right), \quad t\ge 0, \boldsymbol{v}\in\mathbb{R}^{n-1}, u>0.$$

Denote further

$$R_{\zeta_u}(t,\boldsymbol{v},t',\boldsymbol{v}') := Cov\left(\frac{\zeta_u(t,\boldsymbol{v})}{1+ct^\beta u^{-2}}, \frac{\zeta_u(t',\boldsymbol{v}')}{1+ct'^\beta u^{-2}}\right) = \frac{r_{\zeta_u}(|t-t'|, \boldsymbol{v}-\boldsymbol{v}')}{(1+ct^\beta u^{-2})(1+ct'^\beta u^{-2})}, \quad t,t'\ge 0, \boldsymbol{v},\boldsymbol{v}'\in\mathbb{R}^{n-1}.$$

Using the classical approach (see e.g., Lemma 6.1 in Piterbarg (1996)) we have (set $\mathbf{0} = (0,\ldots,0)\in\mathbb{R}^{n-1}$)

$$\mathbb{P}\left\{\sup_{\substack{t\in[0,S_1]\\ \boldsymbol{v}\in\prod_{i=1}^{n-1}\left[0,u^{-2/\alpha_i}S_2\right]}} \frac{\xi_u(t,\boldsymbol{v})}{1+ct^\beta u^{-2}} > f(u)\right\} = \frac{1}{\sqrt{2\pi}f(u)}\exp\left(-\frac{(f(u))^2}{2}\right)$$

$$\times \int_{-\infty}^{\infty} e^{w-\frac{w^2}{2(f(u))^2}} \mathbb{P}\left\{ \sup_{t\in[0,S_1],\boldsymbol{v}\in[0,S_2]^{n-1}} \frac{\zeta_u(t,\boldsymbol{v})}{1+ct^\beta u^{-2}} > f(u) \Big| \zeta_u(0,\mathbf{0}) = f(u) - \frac{w}{f(u)} \right\} dw. \tag{3.14}$$

Further, it follows that

$$\left\{ \frac{\zeta_u(t,\boldsymbol{v})}{1+ct^\beta u^{-2}} \Big| \left(\zeta_u(0,\mathbf{0}) = f(u) - \frac{w}{f(u)}\right), t\in[0,S_1], \boldsymbol{v}\in[0,S_2]^{n-1} \right\}$$

has the same distribution as

$$\left\{ \frac{\zeta_u(t,\boldsymbol{v})}{1+ct^\beta u^{-2}} - R_{\zeta_u}(t,\boldsymbol{v},0,\mathbf{0})\zeta_u(0,\mathbf{0}) + R_{\zeta_u}(t,\boldsymbol{v},0,\mathbf{0}) \left(f(u) - \frac{w}{f(u)}\right), t\in[0,S_1], \boldsymbol{v}\in[0,S_2]^{n-1} \right\}.$$

Thus, the integrand in (3.14) can be rewritten as

$$\begin{aligned} &\mathbb{P}\left\{ \sup_{t\in[0,S_1],\boldsymbol{v}\in[0,S_2]^{n-1}} \frac{\zeta_u(t,\boldsymbol{v})}{1+ct^\beta u^{-2}} - R_{\zeta_u}(t,\boldsymbol{v},0,\mathbf{0})\zeta_u(0,\mathbf{0}) + R_{\zeta_u}(t,\boldsymbol{v},0,\mathbf{0}) \left(f(u) - \frac{w}{f(u)}\right) > f(u) \right\} \\ &= \mathbb{P}\left\{ \sup_{t\in[0,S_1],\boldsymbol{v}\in[0,S_2]^{n-1}} \varsigma_u(t,\boldsymbol{v}) - (f(u))^2(1-R_{\zeta_u}(t,\boldsymbol{v},0,\mathbf{0})) + w(1-R_{\zeta_u}(t,\boldsymbol{v},0,\mathbf{0})) > w \right\}, \end{aligned}$$

where

$$\varsigma_u(t,\boldsymbol{v}) = f(u)\left(\frac{\zeta_u(t,\boldsymbol{v})}{1+ct^\beta u^{-2}} - R_{\zeta_u}(t,\boldsymbol{v},0,\mathbf{0})\zeta_u(0,\mathbf{0}) \right), \quad t\ge 0, \boldsymbol{v}\in\mathbb{R}^{n-1}, u>0.$$

Next, the following convergence

$$(f(u))^2(1-R_{\zeta_u}(t,\boldsymbol{v},0,\mathbf{0})) - w(1-R_{\zeta_u}(t,\boldsymbol{v},0,\mathbf{0})) \to ct^\beta + D_0 t^{\alpha_0} + \sum_{i=1}^{n-1} D_i v_i^{\alpha_i}, \quad u\to\infty$$

holds for any $w\in\mathbb{R}$ uniformly with respect to $t\in[0,S_1], \boldsymbol{v}\in[0,S_2]^{n-1}$. Furthermore,

$$\mathbb{E}\left\{ \left(\varsigma_u(t,\boldsymbol{v}) - \varsigma_u(t',\boldsymbol{v}')\right)^2 \right\} \to 2D_0|t-t'|^{\alpha_0} + 2\sum_{i=1}^{n-1} D_i|v_i-v_i'|^{\alpha_i}, \quad u\to\infty$$

holds uniformly with respect to $t,t'\in[0,S_1], \boldsymbol{v},\boldsymbol{v}'\in[0,S_2]^{n-1}$. It follows thus that

$$\mathbb{E}\left\{ \left(\varsigma_u(t,\boldsymbol{v}) - \varsigma_u(t',\boldsymbol{v}')\right)^2 \right\} \le \mathbb{Q}\left(|t-t'|^{\alpha_0} + \sum_{i=1}^{n-1} |v_i-v_i'|^{\alpha_i} \right)$$

holds for all u sufficiently large and $(t,\boldsymbol{v}),(t',\boldsymbol{v}')$ in any bounded subset of $[0,\infty)\times\mathbb{R}^{n-1}$. Therefore, the family of the random fields $\{\varsigma_u(t,\boldsymbol{v}), t\in[0,S_1], \boldsymbol{v}\in[0,S_2]^{n-1}\}, u>0$ is tight, and thus it converges weakly to $\{\sqrt{2}B_{\alpha_0}(D_0^{1/\alpha_0}t) + \sqrt{2}\sum_{i=1}^{n-1} B_{\alpha_i}(D_i^{1/\alpha_i}v_i), t\in[0,S_1], \boldsymbol{v}\in[0,S_2]^{n-1}\}$ as $u\to\infty$, where $B_{\alpha_i}, i=0,\cdots,n-1$ are independent fBm's with Hurst indexes $\alpha_i/2$, respectively. Further using similar arguments as in Lemma 6.1 of Piterbarg (1996) (see also Michna (2009)) we can show that the limit (letting $u\to\infty$) can be passed under the integral sign in (3.14), and thus the proof is complete. □

Hereafter the diameter of a set $\boldsymbol{A}\subset\mathbb{R}^n, n\in\mathbb{N}$ is defined by

$$\mathrm{diam}(\boldsymbol{A}) = \sup_{\boldsymbol{t},\boldsymbol{s}\in\boldsymbol{A}} ||\boldsymbol{t}-\boldsymbol{s}||,$$

where $||\cdot||$ is the Euclidean norm in $\mathbb{R}^n$. We write $V_n(\boldsymbol{A})$ for the n-dimensional volume of $\boldsymbol{A}$.

Theorem 3.2 *Assume that the conditions of Theorem 3.1 are satisfied. Then there exists some small* $\delta_0>0$ *such that for any* $\boldsymbol{A}\subset\mathbb{R}^{n-1}, n\ge 2$, *with positive volume* $V_{n-1}(\boldsymbol{A})$

$$\begin{aligned} \mathbb{P}\left\{ \sup_{t\in[0,S_1],\boldsymbol{v}\in\boldsymbol{A}} \frac{\xi_u(t,\boldsymbol{v})}{1+ct^\beta u^{-2}} > u \right\} &= V_{n-1}(\boldsymbol{A}) \mathcal{P}_{\alpha_0,\beta}^{cD_0^{-\frac{\beta}{\alpha_0}}} \left[0, D_0^{\frac{1}{\alpha_0}} S_1\right] \prod_{i=1}^{n-1} \mathcal{H}_{\alpha_i} D_i^{\frac{1}{\alpha_i}} \\ &\quad \times \frac{1}{\sqrt{2\pi}} u^{\sum_{i=1}^{n-1}\frac{2}{\alpha_i}-1} \exp\left(-\frac{u^2}{2}\right)(1+o(1)) \end{aligned}$$

holds as $u\to\infty$.

Proof of Theorem 3.2 The proof follows by similar arguments as in the proof of Lemma 7.1 in Piterbarg (1996) or Lemma 6 in Piterbarg (1994b). It is mainly based on the double sum method by splitting the set $\boldsymbol{A}$ into rectangles and then using Bonferroni's inequality with the aid of Theorem 3.1. Since it is lengthy and somehow classical, we shall omit the details. $\square$

3.1 Proof of Theorem 2.3

Set in the following $\delta(u)=\left(\frac{p\ln u}{u}\right)^{1/\beta}, u>0$, with some $p>\max(1/(c\beta),2/(c\alpha))$. First note that, for any sufficiently small $\varepsilon>0$

$$\begin{aligned}\pi_0(u) &:= \mathbb{P}\left\{\sup_{t\in[\delta(u),T]}\Big(\chi_n(t)-g(t)\Big)>u\right\}\\ &\le \mathbb{P}\left\{\sup_{t\in[0,T]}\chi_n(t)>u+(c-\varepsilon)p\frac{\ln u}{u}\right\}\\ &= o\left(u^{n-2+(2/\alpha-1/\beta)_+}\exp\left(-\frac{u^2}{2}\right)\right)\end{aligned}$$

as $u\to\infty$, where the last equality follows from (2) and the Assumption V. Next, we analyze

$$\mathbb{P}\left\{\sup_{t\in[0,\delta(u)]}\Big(\chi_n(t)-g(t)\Big)>u\right\},\quad u\to\infty.$$

Since $\lim_{u\to\infty}\delta(u)=0$ and we are focusing on the asymptotics, by Assumption V the above is asymptotically equivalent with

$$\pi_1(u):=\mathbb{P}\left\{\sup_{t\in[0,\delta(u)]}(\chi_n(t)-ct^\beta)>u\right\},\quad u\to\infty.$$

It follows from our results below that $\pi_0(u)=o(\pi_1(u))$ as $u\to\infty$. The proof is then established by showing further that $\pi_1(u)$ is asymptotically the same as the right-hand side of (2.11). To this end, we need to analyze three cases, namely

i) $\alpha<2\beta$, ii) $\alpha=2\beta$, iii) $\alpha>2\beta$.

Case i) $\alpha<2\beta$: Since $\alpha<2\beta$, for any positive constant S_1, we can divide the interval $[0,\delta(u)]$ into several sub-intervals of length $S_1u^{-2/\alpha}$. Specifically, let for fixed $u>0$

$$\triangle_0=u^{-2/\alpha}[0,S_1],\quad \triangle_k=u^{-2/\alpha}[kS_1,(k+1)S_1],\quad k\in\mathbb{N}.$$

It follows from Bonferroni's inequality that (set $h(u)=\left\lfloor\frac{p^{1/\beta}(\ln u)^{1/\beta}u^{2/\alpha}}{S_1u^{1/\beta}}\right\rfloor+1$)

$$\begin{aligned}\pi_1(u) &\le \sum_{k=0}^{h(u)}\mathbb{P}\left\{\sup_{t\in\triangle_k}(\chi_n(t)-ct^\beta)>u\right\}\\ &\le \sum_{k=0}^{h(u)}\mathbb{P}\left\{\sup_{t\in\triangle_k}\chi_n(t)>u+c(kS_1u^{-2/\alpha})^\beta\right\}\\ &= \sum_{k=0}^{h(u)}\mathbb{P}\left\{\sup_{t\in\triangle_0}\chi_n(t)>u+c(kS_1u^{-2/\alpha})^\beta\right\}=:\pi_2(u).\end{aligned}$$

In view of (2.10)

$$\pi_2(u) = \frac{2^{(2-n)/2}}{\Gamma(n/2)}\mathcal{H}_\alpha[0,S_1]\sum_{k=0}^{h(u)}(u+c(kS_1u^{-2/\alpha})^\beta)^{n-2}\exp\left(-\frac{(u+c(kS_1u^{-2/\alpha})^\beta)^2}{2}\right)(1+o(1))$$

$$
\begin{aligned}
&= \frac{2^{(2-n)/2}}{\Gamma(n/2)} \frac{\mathcal{H}_\alpha[0,S_1]}{S_1} u^{2/\alpha-1/\beta+n-2} \exp\left(-\frac{u^2}{2}\right) \int_0^\infty \exp(-cx^\beta)\, dx (1+o(1)) \\
&= \frac{\Gamma(1/\beta+1)}{c^{1/\beta}} \frac{\mathcal{H}_\alpha[0,S_1]}{S_1} u^{2/\alpha-1/\beta} \Upsilon_n(u)(1+o(1)) \qquad (3.15)
\end{aligned}
$$

as $u \to \infty$, where in the second equality we used the fact that

$$
\lim_{u\to\infty} \sum_{k=0}^{h(u)} \exp\left(-c(kS_1 u^{1/\beta-2/\alpha})^\beta\right) (S_1 u^{1/\beta-2/\alpha}) = \int_0^\infty \exp(-cx^\beta)\, dx.
$$

Similarly, using Bonferroni's inequality we obtain

$$
\pi_1(u) \quad \geq \quad \sum_{k=0}^{h(u)-1} \mathbb{P}\left\{ \sup_{t\in\triangle_k} (\chi_n(t) - ct^\beta) > u \right\} - \Sigma_\chi(u),
$$

where

$$
\Sigma_\chi(u) := \sum_{0\le k<j\le h(u)-1} \mathbb{P}\left\{ \sup_{t\in\triangle_k} (\chi_n(t) - ct^\beta) > u, \sup_{t\in\triangle_j} (\chi_n(t) - ct^\beta) > u \right\}.
$$

Along the lines of the proof of (3.15), we obtain

$$
\sum_{k=0}^{h(u)-1} \mathbb{P}\left\{ \sup_{t\in\triangle_k} (\chi_n(t) - ct^\beta) > u \right\} \quad \geq \quad \frac{\Gamma(1/\beta+1)}{c^{1/\beta}} \frac{\mathcal{H}_\alpha[0,S_1]}{S_1} u^{2/\alpha-1/\beta} \Upsilon_n(u)(1+o(1)) \qquad (3.16)
$$

as $u \to \infty$. Furthermore, we have

$$
\limsup_{S_1\to\infty} \limsup_{u\to\infty} \frac{\Sigma_\chi(u)}{A_2(u)} = 0, \quad \text{with} \ \ A_2(u) := u^{2/\alpha-1/\beta+n-2} \exp\left(-\frac{u^2}{2}\right), \quad u>0. \qquad (3.17)
$$

Consequently, the claim for the case $\alpha < 2\beta$ follows from (3.15)-(3.17). Since the rigorous proof of (3.17) is lengthy, we display it in Appendix.

Case ii) $\alpha = 2\beta$: Clearly, $S_i u^{-2/\alpha} < \delta(u)$ for $S_i > 0, i = 1, 2$, when u is sufficiently large. Hence, we have that

$$
\mathbb{P}\left\{ \sup_{t\in[0,S_2u^{-2/\alpha}]} (\chi_n(t) - ct^\beta) > u \right\} \le \pi_1(u) \le \mathbb{P}\left\{ \sup_{t\in\triangle_0} (\chi_n(t) - ct^\beta) > u \right\} + \sum_{k=1}^{h(u)} \mathbb{P}\left\{ \sup_{t\in\triangle_k} (\chi_n(t) - ct^\beta) > u \right\}.
$$

We give next a technical lemma.

Lemma 3.3 *Assume that $\alpha = 2\beta$. We have*

$$
\mathbb{P}\left\{ \sup_{t\in\triangle_0} (\chi_n(t) - ct^\beta) > u \right\} = \mathcal{P}^c_{\alpha,\beta}[0,S_1]\, \Upsilon_n(u)(1+o(1))
$$

as $u \to \infty$.

Proof of Lemma 3.3 Since $\alpha = 2\beta$, by a time scaling we see that for $u > 0$

$$
\begin{aligned}
\mathbb{P}\left\{ \sup_{t\in\triangle_0} (\chi_n(t) - ct^\beta) > u \right\} &= \mathbb{P}\left\{ \sup_{t\in[0,S_1]} \left(\chi_n(tu^{-2/\alpha}) - ct^\beta u^{-1} > u\right) \right\} \\
&= \mathbb{P}\left\{ \sup_{t\in[0,S_1]} \frac{\chi_n(tu^{-2/\alpha})}{1+ct^\beta u^{-2}} > u \right\}. \qquad (3.18)
\end{aligned}
$$

Introduce the Gaussian random field

$$
Y(t,\boldsymbol{s}) = \sum_{i=1}^n s_i X_i(t), \quad t \ge 0, \boldsymbol{s} = (s_1, \cdots, s_n) \in \mathbb{R}^n.
$$

In the light of Piterbarg (1996) (see page 155 or page 139)

$$\sup_{t\in[0,S_1]} \chi_n(t) = \sup_{(t,\boldsymbol{s})\in\mathcal{G}_{S_1}} Y(t,\boldsymbol{s}),$$

where $\mathcal{G}_{S_1}=[0,S_1]\times\mathcal{S}_{n-1}$, with $\mathcal{S}_{n-1}$ being the unit sphere (with respect to L_2-norm) in $\mathbb{R}^n$. Therefore, continuing (3.18) for $u>0$ we have

$$\mathbb{P}\left\{\sup_{t\in\triangle_0}(\chi_n(t)-ct^\beta)>u\right\} \quad=\quad \mathbb{P}\left\{\sup_{(t,\boldsymbol{s})\in\mathcal{G}_{S_1}}\eta_u(t,\boldsymbol{s})>u\right\}, \tag{3.19}$$

where

$$\eta_u(t,\boldsymbol{s}):=\frac{Y(tu^{-2/\alpha},\boldsymbol{s})}{1+ct^\beta u^{-2}},\quad t\ge 0,\ \ \boldsymbol{s}\in\mathbb{R}^n.$$

Further, it follows that

$$Var(\eta_u(t,\boldsymbol{s}))=\left(\frac{1}{1+ct^\beta u^{-2}}\right)^2,\quad t\ge 0,\ \ \boldsymbol{s}\in\mathcal{S}_{n-1},u>0$$

and, for $t,t'\ge 0,\boldsymbol{s},\boldsymbol{s'}\in\mathcal{S}_{n-1}$

$$Corr(\eta_u(t,\boldsymbol{s}),\eta_u(t',\boldsymbol{s'}))=1-(1-r(u^{-2/\alpha}|t-t'|))-\frac{1}{2}r(u^{-2/\alpha}|t-t'|)||\boldsymbol{s}-\boldsymbol{s'}||^2.$$

We split the sphere $\mathcal{S}_{n-1}$ into sets of small diameters $\{\partial\mathcal{O}_i,0\le i\le\mathcal{Q}\}$, where

$$\mathcal{Q}=\sharp\{\partial\mathcal{O}_i\}<\infty.$$

Note that when $n=1$ the sphere $\mathcal{S}_0$ consists of two points $\{1,-1\}$, and thus in this case the partition $\{\partial\mathcal{O}_i,0\le i\le 1\}$ consists of two single points. The assertions below are valid for this case as well. We have by Bonferroni's inequality

$$\begin{aligned}\sum_{0\le i\le\mathcal{Q}}\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{s}\in\partial\mathcal{O}_i}\eta_u(t,\boldsymbol{s})>u\right\}&\ge\mathbb{P}\left\{\sup_{(t,\boldsymbol{s})\in\mathcal{G}_{S_1}}\eta_u(t,\boldsymbol{s})>u\right\}\\ &\ge\sum_{0\le i\le\mathcal{Q}}\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{s}\in\partial\mathcal{O}_i}\eta_u(t,\boldsymbol{s})>u\right\}-\sum_{0\le i<l\le\mathcal{Q}}\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{s}\in\partial\mathcal{O}_i}\eta_u(t,\boldsymbol{s})>u,\ \sup_{t\in[0,S_1],\boldsymbol{s}\in\partial\mathcal{O}_l}\eta_u(t,\boldsymbol{s})>u\right\}.\end{aligned}$$

We focus next on $\partial\mathcal{O}_0$ which includes $(1,0,\cdots,0)$. When $\mathrm{diam}(\partial\mathcal{O}_0)$ is small enough, we can find a one-to-one projection g from $\partial\mathcal{O}_0$ to the corresponding points where the first component is 1, i.e., $g\boldsymbol{v}=(1,v_2,\cdots,v_n)$ for all $\boldsymbol{v}=(v_1,v_2,\cdots,v_n)\in\partial O_0$. Thus

$$\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{s}\in\partial\mathcal{O}_0}\eta_u(t,\boldsymbol{s})>u\right\}=\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{v}\in g\partial\mathcal{O}_0}\eta_u(t,\boldsymbol{v})>u\right\}.$$

Further, in the light of Lemma 10 of Piterbarg (1994b) we have that, for any $\varepsilon>0$ small enough there exist positive constants δ,u_0 such that, for $\mathrm{diam}(\partial\mathcal{O}_0)<\delta$ and $u>u_0$

$$\begin{aligned}1-\left(1-\frac{\varepsilon}{2}\right)u^{-2}|t-t'|^\alpha-\left(\frac12-\frac{\varepsilon}{4}\right)\sum_{i=2}^n|s_i-s_i'|^2 &\ge Corr(\eta_u(t,\boldsymbol{s}),\eta_u(t',\boldsymbol{s'}))\\ &\ge 1-\left(1+\frac{\varepsilon}{2}\right)u^{-2}|t-t'|^\alpha-\left(\frac12+\frac{\varepsilon}{4}\right)\sum_{i=2}^n|s_i-s_i'|^2\end{aligned}$$

uniformly in $t,t'\ge 0,\boldsymbol{s},\boldsymbol{s'}\in\partial\mathcal{O}_0$. Define two centered stationary Gaussian processes $\{\xi_u^\pm(t,\boldsymbol{v}),t\ge 0,\boldsymbol{v}\in\mathbb{R}^{n-1}\}$ with covariance functions given by (set $\varepsilon_\pm=1\pm\varepsilon$)

$$r_{\xi_u^\pm}(t,\boldsymbol{v})=\exp\left(-\varepsilon_\pm u^{-2}t^\alpha-\frac{\varepsilon_\pm}{2}\sum_{i=1}^{n-1}v_i^2\right),\quad t\ge 0,\boldsymbol{v}\in\mathbb{R}^{n-1},$$

respectively. In view of Slepian's Lemma (see e.g., Falk et al. (2010)) we have

$$\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{v}\in g\partial\mathcal{O}_0}\frac{\xi_u^-(t,\boldsymbol{v})}{1+ct^\beta u^{-2}}>u\right\}\le\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{v}\in g\partial\mathcal{O}_0}\eta_u(t,\boldsymbol{v})>u\right\}\le\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{v}\in g\partial\mathcal{O}_0}\frac{\xi_u^+(t,\boldsymbol{v})}{1+ct^\beta u^{-2}}>u\right\}.$$

Recall at this point that $V_n(\boldsymbol{A})$ denotes the n-dimensional volume of $\boldsymbol{A}\subset\mathbb{R}^n$. Applying Theorem 3.2 to both sides of the last inequality we conclude that

$$V_{n-1}(g\partial\mathcal{O}_0)\mathcal{P}_{\alpha,\beta}^{c(\varepsilon_-)^{-\frac{\beta}{\alpha}}}\left[0,(\varepsilon_-)^{\frac{1}{\alpha}}S_1\right]\varepsilon_-^{\frac{n-1}{2}}\frac{1}{(2\pi)^{n/2}}u^{n-2}\exp\left(-\frac{u^2}{2}\right)(1+o(1))\le\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{v}\in g\partial\mathcal{O}_0}\eta_u(t,\boldsymbol{v})>u\right\}$$
$$\le V_{n-1}(g\partial\mathcal{O}_0)\mathcal{P}_{\alpha,\beta}^{c(\varepsilon_+)^{-\frac{\beta}{\alpha}}}\left[0,(\varepsilon_+)^{\frac{1}{\alpha}}S_1\right]\varepsilon_+^{\frac{n-1}{2}}\frac{1}{(2\pi)^{n/2}}u^{n-2}\exp\left(-\frac{u^2}{2}\right)(1+o(1))$$

as $u\to\infty$, where we used the fact that $\mathcal{H}_2=1/\sqrt{\pi}$. Note that for any sufficiently small positive ε_1, when $\min_{0\le i\le\mathcal{Q}}\operatorname{diam}(\partial\mathcal{O}_i)$ is chosen sufficiently small, we have

$$V_{n-1}(g\partial\mathcal{O}_i)(1-\varepsilon_1)\le V_{n-1}(\partial\mathcal{O}_i)\le V_{n-1}(g\partial\mathcal{O}_i)(1+\varepsilon_1)$$

for any $0\le i\le\mathcal{Q}$. Consequently, by the stationarity in $\boldsymbol{s}$ of the process $\{\eta_u(t,\boldsymbol{s}),(t,\boldsymbol{s})\in\mathcal{G}_{S_1}\}$, and then letting $\varepsilon,\varepsilon_1\to0$, we conclude that

$$\sum_{0\le i\le\mathcal{Q}}\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{s}\in\partial\mathcal{O}_i}\eta_u(t,\boldsymbol{s})>u\right\}=V_{n-1}(\mathcal{S}_{n-1})\mathcal{P}_{\alpha,\beta}^c[0,S_1]\frac{1}{(2\pi)^{n/2}}u^{n-2}\exp\left(-\frac{u^2}{2}\right)(1+o(1))$$

as $u\to\infty$. Moreover, using similar argumentations as in Appendix we show that

$$\sum_{0\le i<l\le\mathcal{Q}}\mathbb{P}\left\{\sup_{t\in[0,S_1],\boldsymbol{s}\in\partial\mathcal{O}_i}\eta_u(t,\boldsymbol{s})>u,\sup_{t\in[0,S_1],\boldsymbol{s}\in\partial\mathcal{O}_l}\eta_u(t,\boldsymbol{s})>u\right\}=o\left(u^{n-2}\exp\left(-\frac{u^2}{2}\right)\right)$$

as $u\to\infty$, and $S_1\to\infty$. Since $V_{n-1}(\mathcal{S}_{n-1})=2\pi^{n/2}/\Gamma(n/2)$ the proof is complete. □

Furthermore, we obtain the following asymptotic upper bound

$$\begin{aligned}\sum_{k=1}^{h(u)}\mathbb{P}\left\{\sup_{t\in\triangle_k}(\chi_n(t)-ct^\beta)>u\right\}&\le\sum_{k=1}^{\infty}\mathbb{P}\left\{\sup_{t\in\triangle_k}\chi_n(t)>u+c(kS_1u^{-2/\alpha})^\beta\right\}\\&\overset{(2.10)}{\le}\mathbb{Q}\,S_1u^{n-2}\exp\left(-\frac{u^2}{2}\right)\sum_{k=1}^{\infty}e^{-c(kS_1)^\beta}(1+o(1))\end{aligned}\tag{3.20}$$

as $u\to\infty$, which together with Lemma 3.3 yields that, for $S_2>0$

$$\begin{aligned}\mathcal{P}_{\alpha,\beta}^c[0,S_2]&\le\liminf_{u\to\infty}\frac{\pi_1(u)}{\Upsilon_n(u)}\\&\le\limsup_{u\to\infty}\frac{\pi_1(u)}{\Upsilon_n(u)}\le\mathcal{P}_{\alpha,\beta}^c[0,S_1]+\mathbb{Q}S_1\sum_{k=1}^{\infty}e^{-c(kS_1)^\beta}.\end{aligned}\tag{3.21}$$

Letting $S_2\to\infty$, we have the finiteness of the *generalized Piterbarg constant*, i.e., $\mathcal{P}_{\alpha,\alpha/2}^c<\infty$. Similarly, letting $S_1\to\infty$ we obtain $\mathcal{P}_{\alpha,\alpha/2}^c>0$. Consequently, the claim for the case $\alpha=2\beta$ follows by letting $S_1,S_2\to\infty$.

Case iii) $\alpha>2\beta$: The lower bound follows immediately since

$$\pi_1(u)\ \ge\ \mathbb{P}\{\chi_n(0)>u\}=\Upsilon_n(u)(1+o(1)).$$

In view of Lemma 3.3 we derive an upper bound as follows

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left\{\sup_{t\in[0,\delta(u)]}(\chi_n(t)-ct^\beta)>u\right\}}{\Upsilon_n(u)}\le\limsup_{u\to\infty}\frac{\mathbb{P}\left\{\sup_{t\in\triangle_0}(\chi_n(t)-ct^\beta)>u\right\}}{\Upsilon_n(u)}=\mathcal{P}_{\alpha,\frac{\alpha}{2}}^c[0,S_1].$$

The proof is completed by letting $S_1\to0$.

3.2 Proof of Theorem 2.5

In this subsection, we give the skeleton of the proof of Theorem 2.5 which is based on the double sum method. Again, we introduce a Gaussian random field

$$Y(t,\boldsymbol{s}) = \sum_{i=1}^{n} s_i X_i(t), \quad t \geq 0, \boldsymbol{s} = (s_1, \cdots, s_n) \in \mathbb{R}^n.$$

Since

$$\sup_{t\in[T_1,T]} \chi_n(t) = \sup_{(t,\boldsymbol{s})\in[T_1,T]\times\mathcal{S}_{n-1}} Y(t,\boldsymbol{s})$$

for any $T_1 \in [0,T)$. For $t,s \geq 0, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{S}_{n-1}$

$$Var(Y(t,\boldsymbol{v})) = \sigma_X^2(t), \text{ and } Corr(Y(t,\boldsymbol{v}), Y(s,\boldsymbol{w})) = r_X(s,t) - \frac{1}{2} r_X(s,t)||\boldsymbol{v}-\boldsymbol{w}||^2.$$

Consequently, by (1.4)–(1.6) there is some $\delta \in (0,T)$ close to T such that

$$Var(Y(t,\boldsymbol{v})) \leq 1 - Aq^{\mu}(u), \quad \text{with } q(u) = \left(\frac{\ln u}{u}\right)^{2/\mu}$$

holds for all $t \in [\delta, T-q(u)]$ and $\boldsymbol{v} \in \mathcal{S}_{n-1}$ when u is sufficiently large, and further for $t,s \in [\delta,T]$ and $\boldsymbol{v},\boldsymbol{w} \in \mathcal{S}_{n-1}$

$$\mathbb{E}\left\{(Y(t,\boldsymbol{v}) - Y(s,\boldsymbol{w}))^2\right\} \leq \mathbb{Q}\left(|t-s|^{\gamma} + ||\boldsymbol{v}-\boldsymbol{w}||^2\right).$$

Therefore, by Piterbarg inequality (cf. Theorem 8.1 of Piterbarg (1996) or Theorem 8.1 in the seminal paper Piterbarg (2001))

$$\Pi_1(u) := \mathbb{P}\left\{\sup_{t\in[\delta,T-q(u)]} \chi_n(t) > u\right\} \leq \mathbb{Q}\, u^{2/\gamma+n-1} \exp\left(\frac{-u^2}{2\left(1 - Aq^{\mu}(u)\right)}\right). \tag{3.22}$$

Furthermore, we have from the Borell-TIS inequality (e.g., Theorem 2.1.1 in Adler and Taylor (2007))

$$\begin{aligned} \Pi_2(u) := \mathbb{P}\left\{\sup_{t\in[T_1,\delta]} \chi_n(t) > u\right\} &\leq \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\mathcal{G}_\delta} Y(t,\boldsymbol{v}) > u\right\} \\ &\leq \exp\left(-\frac{(u-\mathbb{C})^2}{2\sigma_X^2(\delta)}\right). \end{aligned} \tag{3.23}$$

Next, we focus on the asymptotics of

$$\Pi_3(u) := \mathbb{P}\left\{\sup_{t\in[T-q(u),T]} \chi_n(t) > u\right\} = \mathbb{P}\left\{\sup_{t\in[0,q(u)]} \widetilde{\chi}_n(t) > u\right\}, \quad u \to \infty,$$

where $\widetilde{\chi}_n(t) = \chi_n(T-t)$, for $t \in [0, q(u)]$. From the results below we conclude that

$$\Pi_1(u) = o\left(\Pi_3(u)\right), \quad \Pi_2(u) = o\left(\Pi_3(u)\right) \tag{3.24}$$

as $u \to \infty$. The proof is thus established by showing further that $\Pi_3(u)$ is asymptotically the same as the right-hand side of (2.12).

Similar to the proof of Theorem 2.3 we need to distinguish between the following three cases:

i) $\nu < \mu$, ii) $\nu = \mu$, iii) $\nu > \mu$.

Let, for $S_1 > 0$

$$\triangle_0 = u^{-2/\nu}[0, S_1], \quad \triangle_k = u^{-2/\nu}[kS_1, (k+1)S_1], \ k \in \mathbb{N},$$

and define $\theta(u) = \left\lfloor \frac{(\ln u)^{2/\mu} u^{2/\nu}}{S_1 u^{2/\mu}} \right\rfloor + 1$.

Case i) $\nu < \mu$: Since $\nu < \mu$, using Bonferroni's inequality, we have

$$\begin{aligned}\sum_{k=0}^{\theta(u)} \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_k\times\mathcal{S}_{n-1}} Z(t,\boldsymbol{v})>u\right\} &= \sum_{k=0}^{\theta(u)} \mathbb{P}\left\{\sup_{t\in\triangle_k}\widetilde{\chi}_n(t)>u\right\}\\ &\geq \mathbb{P}\left\{\sup_{t\in[0,q(u)]}\widetilde{\chi}_n(t)>u\right\}\\ &\geq \sum_{k=0}^{\theta(u)-1} \mathbb{P}\left\{\sup_{t\in\triangle_k}\widetilde{\chi}_n(t)>u\right\}-\Sigma_{\widetilde{\chi}_n}(u),\end{aligned}$$

where $Z(t,\boldsymbol{v})=Y(T-t,\boldsymbol{v})$, for $(t,\boldsymbol{v})\in[0,q(u)]\times\mathcal{S}_{n-1}$, and

$$\Sigma_{\widetilde{\chi}_n}(u):=\sum_{0\leq k<j\leq\theta(u)-1}\mathbb{P}\left\{\sup_{t\in\triangle_k}\widetilde{\chi}_n(t)>u,\ \sup_{t\in\triangle_j}\widetilde{\chi}_n(t)>u\right\}.$$

For any $\varepsilon\in(0,1)$ and all u large

$$1-A(1-\varepsilon)t^{\mu}>Var(Z(t,\boldsymbol{v}))^{1/2}>1-A(1+\varepsilon)t^{\mu}$$

and

$$\begin{aligned}2D(1-\varepsilon)|t-s|^{\nu}+(1-\varepsilon)||\boldsymbol{v}-\boldsymbol{w}||^2 &\leq \mathbb{E}\left\{(Z(t,\boldsymbol{v})-Z(s,\boldsymbol{w}))^2\right\}\\ &\leq 2D(1+\varepsilon)|t-s|^{\nu}+(1+\varepsilon)||\boldsymbol{v}-\boldsymbol{w}||^2.\end{aligned}$$

Next we introduce a centered stationary Gaussian process $\{\xi(t),t\geq0\}$ with covariance function

$$r_{\xi}(t)=\exp\left(-Dt^{\nu}\right),\quad t\geq0$$

and set

$$Z_2(t,\boldsymbol{v})=\sum_{i=1}^{n}v_i\xi_i(t),\quad t\geq0,\ \boldsymbol{v}\in\mathbb{R}^n,$$

with $\{\xi_i(t),t\geq0\},1\leq i\leq n$, being independent copies of $\{\xi(t),t\geq0\}$. Thus for $(t,\boldsymbol{v})\in[0,q(u)]\times\mathcal{S}_{n-1}$ and all u large

$$\begin{aligned}2D(1-\varepsilon)|t-s|^{\nu}+(1-\varepsilon)||\boldsymbol{v}-\boldsymbol{w}||^2 &\leq \mathbb{E}\left\{(Z_2(t,\boldsymbol{v})-Z_2(s,\boldsymbol{w}))^2\right\}\\ &\leq 2D(1+\varepsilon)|t-s|^{\nu}+(1+\varepsilon)||\boldsymbol{v}-\boldsymbol{w}||^2.\end{aligned}$$

Since ε can be arbitrary small, using Slepian's Lemma we conclude that

$$\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_k\times\mathcal{S}_{n-1}}Z(t,\boldsymbol{v})>u\right\}=\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_k\times\mathcal{S}_{n-1}}Z_2(t,\boldsymbol{v})(1-At^{\mu})>u\right\}(1+o(1))\tag{3.25}$$

as $u\to\infty$. Consequently, as $u\to\infty$

$$\sum_{k=0}^{\theta(u)}\mathbb{P}\left\{\sup_{t\in\triangle_k}\widetilde{\chi}_n(t)>u\right\}\leq\sum_{k=0}^{\theta(u)}\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_k\times\mathcal{S}_{n-1}}Z_2(t,\boldsymbol{v})>\frac{u}{1-A(kS_1u^{-2/\nu})^{\mu}}\right\}(1+o(1))=:\pi_3(u).$$

Utilising further (2.10), we obtain

$$\begin{aligned}\pi_3(u) &= \frac{2^{(2-n)/2}}{\Gamma(n/2)}\mathcal{H}_{\nu}[0,D^{1/\nu}S_1]\sum_{k=0}^{\theta(u)}\left(\frac{u}{1-A(kS_1u^{-2/\nu})^{\mu}}\right)^{n-2}\exp\left(-\frac{u^2\left(1+A(kS_1u^{-2/\nu})^{\mu}\right)^2}{2}\right)(1+o(1))\\ &= \frac{\mathcal{H}_{\nu}[0,D^{1/\nu}S_1]}{S_1}u^{2/\nu-2/\mu}\Upsilon_n(u)\int_0^{\infty}\exp\left(-Ax^{\mu}\right)dx(1+o(1))\end{aligned}$$

$$= \quad D^{1/\nu}\frac{\Gamma(1/\mu+1)}{A^{1/\mu}}\frac{\mathcal{H}_\nu[0,D^{1/\nu}S_1]}{D^{1/\nu}S_1}u^{2/\nu-2/\mu}\Upsilon_n(u)(1+o(1)) \tag{3.26}$$

as $u\to\infty$. Using the same argumentations as (3.26) the following asymptotic lower bound

$$\sum_{k=0}^{\theta(u)-1}\mathbb{P}\left\{\sup_{t\in\triangle_k}\widetilde{\chi}_n(t)>u\right\}\ge D^{1/\nu}\frac{\Gamma(1/\mu+1)}{A^{1/\mu}}\frac{\mathcal{H}_\nu[0,D^{1/\nu}S_1]}{D^{1/\nu}S_1}u^{2/\nu-2/\mu}\Upsilon_n(u)(1+o(1)) \tag{3.27}$$

holds as $u\to\infty$. Furthermore, we have

$$\limsup_{S_1\to\infty}\limsup_{u\to\infty}\frac{\Sigma_{\widetilde{\chi}_n}(u)}{A_3(u)}=0,\quad \text{with}\quad A_3(u):=u^{2/\nu-2/\mu+n-2}\exp\left(-\frac{u^2}{2}\right),\quad u>0 \tag{3.28}$$

the proof of which is omitted since it is similar to (3.17). Consequently, the claim for the case $\nu<\mu$ follows from (3.26)-(3.28).

Case ii) $\nu=\mu$: Since $S_iu^{-2/\nu}<q(u)=\left(\frac{\ln u}{u}\right)^{2/\mu}$ for $S_i>0, i=1,2$, when u is sufficiently large. Hence, we have that

$$\mathbb{P}\left\{\sup_{t\in[0,S_2u^{-2/\nu}]}\widetilde{\chi}_n(t)>u\right\}\le\pi_1(u)\le\mathbb{P}\left\{\sup_{t\in\triangle_0}\widetilde{\chi}_n(t)>u\right\}+\sum_{k=1}^{\theta(u)}\mathbb{P}\left\{\sup_{t\in\triangle_k}\widetilde{\chi}_n(t)>u\right\}.$$

From (3.25) we obtain further

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in\triangle_0}\widetilde{\chi}_n(t)>u\right\} &= \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_0\times\mathcal{S}_{n-1}}Z_2(t,\boldsymbol{v})(1-At^\mu)>u\right\}(1+o(1))\\ &= \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_0\times\mathcal{S}_{n-1}}\frac{Z_2(t,\boldsymbol{v})}{(1+At^\mu)}>u\right\}(1+o(1))\end{aligned}$$

as $u\to\infty$. In view of Theorem 3.1 and Theorem 3.2, and the derivation of the case $\alpha=2\beta$ in the last subsection, we conclude that

$$\mathbb{P}\left\{\sup_{t\in\triangle_0}\widetilde{\chi}_n(t)>u\right\} = \mathcal{P}_{\nu,\mu}^{AD^{-1}}[0,D^{\frac{1}{\nu}}S_1]\Upsilon_n(u)(1+o(1)) \tag{3.29}$$

as $u\to\infty$. Now, the claim follows using the same argumentation as (3.21).

Case iii) $\nu>\mu$: By (3.29) the upper bound is derived as

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left\{\sup_{t\in[0,q(u)]}\widetilde{\chi}_n(t)>u\right\}}{\Upsilon_n(u)} \le \limsup_{u\to\infty}\frac{\mathbb{P}\left\{\sup_{t\in\triangle_0}\widetilde{\chi}_n(t)>u\right\}}{\Upsilon_n(u)}\le\mathcal{P}_{\nu,\nu}^{AD^{-1}}[0,D^{\frac{1}{\nu}}S_1].$$

Since further

$$\mathbb{P}\left\{\sup_{t\in[0,q(u)]}\widetilde{\chi}_n(t)>u\right\} \ge \mathbb{P}\left\{\widetilde{\chi}_n(0)>u\right\}=\Upsilon_n(u)(1+o(1))$$

the proof of this case is established by letting $S_1\to0$. Consequently, it follows that (3.24) is valid, and thus the proof is complete. □

3.3 Proof of Theorem 2.6

For $\delta\in(0,T)$, set

$$\Pi(u):=\mathbb{P}\left\{\sup_{t\in[\delta,T]}\Big(\chi_n(t)-g(t)\Big)>u\right\}.$$

Thus for any $u\ge0$

$$\Pi(u)\le\mathbb{P}\left\{\sup_{t\in[0,T]}\Big(\chi_n(t)-g(t)\Big)>u\right\}\le\mathbb{P}\left\{\sup_{t\in[0,\delta]}\Big(\chi_n(t)-g(t)\Big)>u\right\}+\Pi(u). \tag{3.30}$$

It follows that

$$\begin{aligned}\Pi(u) &= \mathbb{P}\left\{\sup_{t\in[\delta,T]}\frac{\chi_n(t)}{u+g(t)}>1\right\}\\ &= \mathbb{P}\left\{\sup_{t\in[\delta,T]}\frac{\chi_n(t)}{\sigma_X(t)}\frac{m_u(T)}{m_u(t)}>m_u(T)\right\},\quad \text{with } m_u(t):=\frac{u+g(t)}{\sigma_X(t)},\ t\ge 0.\end{aligned}$$

For any $t\in[0,T]$

$$1-\frac{m_u(T)}{m_u(t)}=\frac{\sigma_X(T)-\sigma_X(t)}{\sigma_X(T)}+\frac{\sigma_X(t)(g(t)-g(T))}{(u+g(t))\sigma_X(T)}.$$

Further, in view of (1.4) and Assumption VI, and noting that $\mu\le\widetilde{\beta}$, δ can be chosen close enough to T such that

$$|g(T)-g(t)|\le\mathbb{Q}\left(\sigma_X(T)-\sigma_X(t)\right)$$

for all $t\in[\delta,T]$. Hence for any $\varepsilon\in(0,1)$, when u is sufficiently large, we have, uniformly in $[\delta,T]$

$$1-(1+\varepsilon)\frac{\sigma_X(T)-\sigma_X(t)}{\sigma_X(T)}\le\frac{m_u(T)}{m_u(t)}\le 1-(1-\varepsilon)\frac{\sigma_X(T)-\sigma_X(t)}{\sigma_X(T)}. \tag{3.31}$$

Therefore, for u sufficiently large

$$\pi_{+\varepsilon}(u):=\mathbb{P}\left\{\sup_{t\in[\delta,T]}Y_{+\varepsilon}(t)>m_u(T)\right\}\le\Pi(u)\le\pi_{-\varepsilon}(u):=\mathbb{P}\left\{\sup_{t\in[\delta,T]}Y_{-\varepsilon}(t)>m_u(T)\right\},$$

where

$$Y_{\pm\varepsilon}(t)=\sqrt{\sum_{i=1}^n Y_{\pm\varepsilon,i}^2(t)},\quad t\ge 0,$$

with

$$Y_{\pm\varepsilon,i}(t):=\frac{X_i(t)}{\sigma_X(t)}\left(1-(1\pm\varepsilon)\frac{\sigma_X(T)-\sigma_X(t)}{\sigma_X(T)}\right),\quad t\ge 0,\ 1\le i\le n.$$

Since the analysis of $\pi_{+\varepsilon}(u)$ and $\pi_{-\varepsilon}(u)$ are the same, next we only discuss $\pi_{+\varepsilon}(u)$ for fixed $\varepsilon\in(0,1)$. The variance function $\sigma_Y(t)$ of $Y_{+\varepsilon,1}(t)$ attains its maximum over $[\delta,T]$ at the unique point T with

$$\sigma_Y(t)=1-A(1+\varepsilon)(T-t)^{\mu}(1+o(1)),\quad \text{as } t\to T.$$

Further, by (1.5)

$$r_Y(s,t)=Corr(Y_{+\varepsilon,1}(s),Y_{+\varepsilon,1}(t))=1-D|t-s|^{\nu}+o(|t-s|^{\nu}),\quad t,s\uparrow T. \tag{3.32}$$

Moreover, in view of Assumption IV for $s,t\in[\delta,T]$

$$\mathbb{E}\left\{(Y_{+\varepsilon,1}(t)-Y_{+\varepsilon,1}(s))^2\right\}\quad\le\quad\mathbb{Q}\,|s-t|^{\gamma}.$$

Consequently, by Theorem 2.5

$$\pi_{+\varepsilon}(u)=\mathcal{W}_{\nu,\mu}^{\varepsilon}\left(m_u(T)\right)^{(2/\nu-2/\mu)_+}\Upsilon_n\left(m_u(T)\right)(1+o(1)),\quad u\to\infty,$$

where

$$\mathcal{W}_{\nu,\mu}^{\varepsilon}=\begin{cases}D^{1/\nu}\frac{\Gamma(1/\mu+1)}{((1+\varepsilon)A)^{1/\mu}}\mathcal{H}_{\nu}, & \text{if } \nu<\mu,\\ \mathcal{P}_{\nu,\mu}^{A(1+\varepsilon)D^{-1}}, & \text{if } \nu=\mu,\\ 1 & \text{if } \nu>\mu.\end{cases}$$

Letting $\varepsilon\to 0$, we conclude that

$$\Pi(u)=\mathcal{W}_{\nu,\mu}^{0}\left(m_u(T)\right)^{(2/\nu-2/\mu)_+}\Upsilon_n\left(m_u(T)\right)(1+o(1))$$

as $u\to\infty$. In addition, by the Borell-TIS inequality, for u sufficiently large

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[0,\delta]}\Big(\chi_n(t)-g(t)\Big)>u\right\} &\le \mathbb{P}\left\{\sup_{t\in[0,\delta]}\chi_n(t)>u\right\}\\
&= \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\mathcal{G}_\delta} Y(t,\boldsymbol{v})>u\right\}\\
&\le \exp\left(-\frac{(u-\mathbb{C})^2}{2\sigma_X^2(\delta)}\right)
\end{aligned}$$

and thus the claim follows from the last two formulas. □

4 Appendix

This section is dedicated to the proof of (3.17). Let

$$B(k,S_1,u)=u+c(kS_1u^{-2/\alpha})^\beta,\quad k\in\mathbb{N},\ S_1>0,\ u>0.$$

The double sum $\Sigma_\chi(u)$ can be divided into two parts, i.e.,

$$\Sigma_\chi(u)=\sum_{0\le k<j\le h(u)-1}\mathbb{P}\left\{\sup_{t\in\triangle_k}(\chi_n(t)-ct^\beta)>u,\ \sup_{t\in\triangle_j}(\chi_n(t)-ct^\beta)>u\right\}=:\Sigma_{\chi,1}(u)+\Sigma_{\chi,2}(u),$$

where $\Sigma_{\chi,1}(u)$ is the sum for $j=k+1$, and $\Sigma_{\chi,2}(u)$ is the sum for $j>k+1$. We first give the estimation of the first sum. It follows that

$$\Sigma_{\chi,1}(u)\ \le\ \sum_{k=0}^{h(u)}\mathbb{P}\left\{\sup_{t\in\triangle_k}\chi_n(t)>B(k,S_1,u),\ \sup_{t\in\triangle_{k+1}}\chi_n(t)>B(k,S_1,u)\right\}. \tag{4.33}$$

Further, we have that

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in\triangle_k}\chi_n(t)>B(k,S_1,u),\ \sup_{t\in\triangle_{k+1}}\chi_n(t)>B(k,S_1,u)\right\}\\
=\ &\mathbb{P}\left\{\sup_{t\in\triangle_k}\chi_n(t)>B(k,S_1,u)\right\}+\mathbb{P}\left\{\sup_{t\in\triangle_{k+1}}\chi_n(t)>B(k,S_1,u)\right\}\\
&-\mathbb{P}\left\{\sup_{t\in\triangle_k\cup\triangle_{k+1}}\chi_n(t)>B(k,S_1,u)\right\},
\end{aligned}$$

which in view of (3.15) implies

$$\lim_{S_1\to\infty}\limsup_{u\to\infty}\frac{\Sigma_{\chi,1}(u)}{A_2(u)}\le\mathbb{Q}\lim_{S_1\to\infty}\frac{2\mathcal{H}_\alpha[0,S_1]-\mathcal{H}_\alpha[0,2S_1]}{S_1}=0.$$

In order to estimate $\Sigma_{\chi,2}(u)$, we introduce a Gaussian random field

$$Y(t,\boldsymbol{v})=\sum_{i=1}^n v_iX_i(t),\quad t\ge0,\ \boldsymbol{v}=(v_1,\cdots,v_n)\in\mathbb{R}^n.$$

In the light of Piterbarg (1996)

$$\sup_{t\in[0,S_1]}\chi_n(t)=\sup_{(t,\boldsymbol{v})\in\mathcal{G}_{S_1}}Y(t,\boldsymbol{v}),$$

where $\mathcal{G}_{S_1}=[0,S_1]\times\mathcal{S}_{n-1}$, with $\mathcal{S}_{n-1}$ being the unit sphere in $\mathbb{R}^n$. Consequently,

$$\Sigma_{\chi,2}(u)\le\sum_{k=0}^{h(u)-1}\sum_{j=k+2}^{h(u)-1}\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_k\times\mathcal{S}_{n-1}}Y(t,\boldsymbol{v})>B(k,S_1,u),\ \sup_{(t,\boldsymbol{v})\in\triangle_j\times\mathcal{S}_{n-1}}Y(t,\boldsymbol{v})>B(k,S_1,u)\right\}. \tag{4.34}$$

We split the sphere $\mathcal{S}_{n-1}$ into sets of small diameters $\{\partial\mathcal{S}_i, 0 \le i \le \mathcal{N}_*\}$, where

$$\mathcal{N}_* = \sharp\{\partial\mathcal{S}_i\} < \infty.$$

Further, we see that the summand on the right-hand side of (4.34) is not greater than $\Theta_1^{k,j}(u) + \Theta_2^{k,j}(u)$, with

$$\begin{aligned}
\Theta_1^{k,j}(u) &= \sum_{\substack{0\le i,l\le \mathcal{N}_* \\ \partial\mathcal{S}_i\cap\partial\mathcal{S}_l=\emptyset}} \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_k\times\partial\mathcal{S}_i} Y(t,\boldsymbol{v}) > B(k,S_1,u), \sup_{(t,\boldsymbol{v})\in\triangle_j\times\partial\mathcal{S}_l} Y(t,\boldsymbol{v}) > B(k,S_1,u)\right\} \\
\Theta_2^{k,j}(u) &= \sum_{\substack{0\le i,l\le \mathcal{N}_* \\ \partial\mathcal{S}_i\cap\partial\mathcal{S}_l\neq\emptyset}} \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_k\times\partial\mathcal{S}_i} Y(t,\boldsymbol{v}) > B(k,S_1,u), \sup_{(t,\boldsymbol{v})\in\triangle_j\times\partial\mathcal{S}_l} Y(t,\boldsymbol{v}) > B(k,S_1,u)\right\},
\end{aligned}$$

where $\partial\mathcal{S}_i\cap\partial\mathcal{S}_l \neq \emptyset$ means $\partial\mathcal{S}_i, \partial\mathcal{S}_l$ are identical or adjacent, and $\partial\mathcal{S}_i\cap\partial\mathcal{S}_l = \emptyset$ means $\partial\mathcal{S}_i, \partial\mathcal{S}_l$ are neither identical nor adjacent. Denote the distance of two sets $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^n, n \in \mathbb{N}$, as

$$\rho(\boldsymbol{A},\boldsymbol{B}) = \inf_{\boldsymbol{x}\in\boldsymbol{A},\boldsymbol{y}\in\boldsymbol{B}} ||\boldsymbol{x}-\boldsymbol{y}||.$$

If $\partial\mathcal{S}_i\cap\partial\mathcal{S}_l = \emptyset$ then there exists some small positive constant ρ_0 (independent of i,l) such that $\rho(\partial\mathcal{S}_i,\partial\mathcal{S}_l) > \rho_0$. Next, we estimate $\Theta_1^{k,j}(u)$. For any $u \ge 0$

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_k\times\partial\mathcal{S}_i} Y(t,\boldsymbol{v}) > B(k,S_1,u), \sup_{(t,\boldsymbol{v})\in\triangle_j\times\partial\mathcal{S}_l} Y(t,\boldsymbol{v}) > B(k,S_1,u)\right\} \\
\le\ & \mathbb{P}\left\{\sup_{\substack{(t,s)\in\triangle_k\times\triangle_j \\ \boldsymbol{v}\in\partial\mathcal{S}_i,\boldsymbol{w}\in\partial\mathcal{S}_l}} Z(t,\boldsymbol{v},s,\boldsymbol{w}) > 2u\right\},
\end{aligned}$$

where

$$Z(t,\boldsymbol{v},s,\boldsymbol{w}) = Y(t,\boldsymbol{v}) + Y(s,\boldsymbol{w}),\ t,s\ge 0,\ \boldsymbol{v},\boldsymbol{w}\in\mathbb{R}^n.$$

When u is sufficiently large for $(t,s)\in\triangle_k\times\triangle_j\subset[0,1]^2, \boldsymbol{v}\in\partial\mathcal{S}_i\subset[-2,2]^n, \boldsymbol{w}\in\partial\mathcal{S}_l\subset[-2,2]^n$, with $\rho(\partial\mathcal{S}_i,\partial\mathcal{S}_l) > \rho_0$ we have

$$\begin{aligned}
Var(Z(t,\boldsymbol{v},s,\boldsymbol{w})) &= 4 - \left(2(1-r(s-t)) + r(s-t)||\boldsymbol{v}-\boldsymbol{w}||^2\right) \\
&\le 4(1-\delta_0),
\end{aligned}$$

for some $\delta_0 > 0$. Therefore, it follows from the Borell-TIS inequality (see e.g., Adler and Taylor (2007)) that

$$\Theta_1^{k,j}(u) \le \mathbb{Q}\mathcal{N}_*\exp\left(-\frac{(u-a)^2}{2(1-\delta_0)}\right),\ \text{with } a = \mathbb{E}\left\{\sup_{\substack{(t,s)\in[0,1]^2 \\ (\boldsymbol{v},\boldsymbol{w})\in[-2,2]^{2n}}} Z(t,\boldsymbol{v},s,\boldsymbol{w})\right\} < \infty.$$

Consequently,

$$\limsup_{u\to\infty}\frac{\sum_{k=0}^{h(u)-1}\sum_{j=k+2}^{h(u)-1}\Theta_1^{k,j}(u)}{A_2(u)} = 0. \tag{4.35}$$

Next, we estimate $\sum_{k=0}^{h(u)-1}\sum_{j=k+2}^{\infty}\Theta_2^{k,j}(u)$. The stationarity of $\{Y(t,\boldsymbol{v}), t\ge 0, \boldsymbol{v}\in\mathcal{S}_{n-1}\}$ implies

$$\sum_{k=0}^{h(u)-1}\sum_{j\ge k+2}\Theta_2^{k,j}(u) \le \mathbb{Q}\sum_{k=0}^{h(u)-1}\sum_{j\ge 2}\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_0\times\partial\mathcal{S}_i} Y(t,\boldsymbol{v}) > B(k,S_1,u), \sup_{(t,\boldsymbol{v})\in\triangle_j\times\partial\mathcal{S}_l} Y(t,\boldsymbol{v}) > B(k,S_1,u)\right\}$$

for some fixed $\partial\mathcal{S}_i, \partial\mathcal{S}_l$ satisfying $\partial\mathcal{S}_i\cap\partial\mathcal{S}_l\neq\emptyset$. Additionally, $\mathrm{diam}(\partial\mathcal{S}_i\cup\partial\mathcal{S}_l)$ can be chosen sufficiently small such that $\partial\mathcal{S}_i, \partial\mathcal{S}_l$ are in $\partial\mathcal{O}_0$, which is a subset of $\mathcal{S}_{n-1}$ and includes $(1,0,\cdots,0)$, and further on $\partial\mathcal{O}_0$ we can find a one-to-one projection g from it to the corresponding points where the first component is 1, i.e., $g\boldsymbol{v} = (1, v_2, \cdots, v_n)$ for all $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \partial\mathcal{O}_0$.

Let

$$\widetilde{\triangle}_0 = \left[0, \frac{S_2}{u}\right]^{n-1} \cap g\partial\mathcal{O}_0,\ \widetilde{\triangle}_{\boldsymbol{k}} = \prod_{i=1}^{n-1}\left[k_i\frac{S_2}{u}, (k_i+1)\frac{S_2}{u}\right] \cap g\partial\mathcal{O}_0,\ \boldsymbol{k} \in \mathbb{Z}^{n-1}$$

and

$$\mathcal{K}_i = \{\boldsymbol{k} : \widetilde{\triangle}_{\boldsymbol{k}} \cap g\partial\mathcal{S}_i \neq \emptyset\}, \quad \mathcal{K}_l = \{\boldsymbol{k} : \widetilde{\triangle}_{\boldsymbol{k}} \cap g\partial\mathcal{S}_l \neq \emptyset\}.$$

With these notation, we have that

$$\begin{aligned}\sum_{k=0}^{h(u)-1}\sum_{j\ge k+2}\Theta_2^{k,j}(u) \quad &\le \quad \mathbb{Q}\sum_{k=0}^{h(u)-1}\sum_{j\ge 2}\sum_{\boldsymbol{i}\in\mathcal{K}_i}\sum_{\boldsymbol{l}\in\mathcal{K}_l}\\ &\qquad \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_0\times\widetilde{\triangle}_{\boldsymbol{i}}} Y(t,\boldsymbol{v}) > B(k,S_1,u), \sup_{(t,\boldsymbol{v})\in\triangle_j\times\widetilde{\triangle}_{\boldsymbol{l}}} Y(t,\boldsymbol{v}) > B(k,S_1,u)\right\}.\end{aligned}$$

The last sums on the right-hand side can be divided into two terms $I_i(u), i = 1,2$, according to whether $\widetilde{\triangle}_{\boldsymbol{i}} \cap \widetilde{\triangle}_{\boldsymbol{l}} \neq \emptyset$ or not. We derive that

$$\begin{aligned}&\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_0\times\widetilde{\triangle}_{\boldsymbol{i}}} Y(t,\boldsymbol{v}) > B(k,S_1,u), \sup_{(t,\boldsymbol{v})\in\triangle_j\times\widetilde{\triangle}_{\boldsymbol{l}}} Y(t,\boldsymbol{v}) > B(k,S_1,u)\right\}\\ \le\quad &\mathbb{P}\left\{\sup_{\substack{(t,s)\in\triangle_0\times\triangle_j\\ \boldsymbol{v}\in\widetilde{\triangle}_{\boldsymbol{i}},\boldsymbol{w}\in\widetilde{\triangle}_{\boldsymbol{l}}}} Z(t,\boldsymbol{v},s,\boldsymbol{w}) > 2B(k,S_1,u)\right\},\end{aligned}$$

where

$$Z(t,\boldsymbol{v},s,\boldsymbol{w}) = Y(t,\boldsymbol{v}) + Y(s,\boldsymbol{w}),\ t,s \ge 0,\ \boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^{n-1}.$$

It follows that, for $(t,s) \in \triangle_0 \times \triangle_j$, $\boldsymbol{v} \in \widetilde{\triangle}_{\boldsymbol{i}}, \boldsymbol{w} \in \widetilde{\triangle}_{\boldsymbol{l}}$, $\mathrm{diam}(\partial\mathcal{O}_0)$ sufficiently small, and u sufficiently large

$$2 \le Var(Z(t,\boldsymbol{v},s,\boldsymbol{w})) \le 4\left(1 - \frac{1}{4}((j-1)S_1)^\alpha u^{-2}\right). \tag{4.36}$$

Further, set $\overline{Z}(t,\boldsymbol{v},s,\boldsymbol{w}) = Z(t,\boldsymbol{v},s,\boldsymbol{w})/\sqrt{Var(Z(t,\boldsymbol{v},s,\boldsymbol{w}))}$. Borrowing the arguments of the proof of Lemma 6.3 in Piterbarg (1996) we show that

$$\mathbb{E}\left\{(\overline{Z}(t,\boldsymbol{v},s,\boldsymbol{w}) - \overline{Z}(t',\boldsymbol{v}',s',\boldsymbol{w}')^2\right\} \le 4\Big(\mathbb{E}\left\{(Y(t,\boldsymbol{v}) - Y(t',\boldsymbol{v}'))^2\right\} + \mathbb{E}\left\{(Y(s,\boldsymbol{w}) - Y(s',\boldsymbol{w}')^2\right\}\Big).$$

Moreover, as in Lemma 10 of Piterbarg (1994b), for $\mathrm{diam}(\partial\mathcal{O}_0)$ sufficiently small, and all u large

$$\mathbb{E}\left\{(Y(t,\boldsymbol{v}) - Y(t',\boldsymbol{v}')^2\right\} \quad \le \quad 4|t-t'|^\alpha + 2\sum_{i=2}^n (v_i - v_i')^2.$$

Therefore

$$\begin{aligned}\mathbb{E}\left\{(\overline{Z}(t,\boldsymbol{v},s,\boldsymbol{w}) - \overline{Z}(t',\boldsymbol{v}',s',\boldsymbol{w}'))^2\right\} \quad &\le \quad 16|t-t'|^\alpha + 16|s-s'|^\alpha + 8\sum_{i=2}^n (v_i - v_i')^2 + 8\sum_{i=2}^n (w_i - w_i')^2\\ &\le \quad 2(1 - r_\zeta(|t-t'|, |s-s'|, \boldsymbol{v}-\boldsymbol{v}', \boldsymbol{w}-\boldsymbol{w}')),\end{aligned} \tag{4.37}$$

where

$$r_\zeta(t,s,\boldsymbol{v},\boldsymbol{w}) = \exp\left(-9t^\alpha - 9s^\alpha - 5\sum_{i=2}^n v_i^2 - 5\sum_{i=2}^n w_i^2\right), \quad t,s \ge 0, \boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^{n-1}$$

is the covariance function of a stationary Gaussian random field $\{\zeta(t,s,\boldsymbol{v},\boldsymbol{w}), t,s\geq 0, \boldsymbol{v},\boldsymbol{w}\in\mathbb{R}^{n-1}\}$. Consequently, in view of (4.36) and (4.37), and thanks to Slepian's Lemma, we obtain

$$
\begin{aligned}
&\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\triangle_0\times\widetilde{\triangle}_{\boldsymbol{i}}} Y(t,\boldsymbol{v})>B(k,S_1,u), \sup_{(t,\boldsymbol{v})\in\triangle_j\times\widetilde{\triangle}_{\boldsymbol{l}}} Y(t,\boldsymbol{v})>B(k,S_1,u)\right\}\\
\leq\ &\mathbb{P}\left\{\sup_{\substack{(t,s)\in\triangle_0\times\triangle_j\\ \boldsymbol{v}\in\widetilde{\triangle}_{\boldsymbol{i}},\boldsymbol{w}\in\widetilde{\triangle}_{\boldsymbol{l}}}}\zeta(t,s,\boldsymbol{v},\boldsymbol{w})>\frac{2B(k,S_1,u)}{\sqrt{4-((j-1)S_1)^{\alpha}u^{-2}}}\right\}.
\end{aligned}
$$

Since, for any cube $\widetilde{\triangle}_{\boldsymbol{i}}$ in $\mathbb{R}^{n-1}$ there are 3^{n-1} cubes having non-empty intersection with it, we have

$$
\begin{aligned}
I_1(u)\ &\leq\ \sum_{k=0}^{h(u)-1}\sum_{j\geq 2}\sum_{\boldsymbol{i}\in\mathcal{K}_i}\sum_{\substack{\boldsymbol{l}\in\mathcal{K}_l\\ \widetilde{\triangle}_{\boldsymbol{i}}\cap\widetilde{\triangle}_{\boldsymbol{l}}\neq\emptyset}}\mathbb{P}\left\{\sup_{\substack{(t,s)\in\triangle_0\times\triangle_j\\ \boldsymbol{v}\in\widetilde{\triangle}_{\boldsymbol{i}},\boldsymbol{w}\in\widetilde{\triangle}_{\boldsymbol{l}}}}\zeta(t,s,\boldsymbol{v},\boldsymbol{w})>\frac{2B(k,S_1,u)}{\sqrt{4-((j-1)S_1)^{\alpha}u^{-2}}}\right\}\\
&\leq\ 3^{n-1}\sum_{k=0}^{h(u)-1}\sum_{j\geq 2}\sum_{\boldsymbol{i}\in\mathcal{K}_i}\mathbb{P}\left\{\sup_{\substack{(t,s)\in\triangle_0\times\triangle_j\\ \boldsymbol{v}\in\widetilde{\triangle}_{\boldsymbol{i}},\boldsymbol{w}\in\widetilde{\triangle}_{\boldsymbol{l}}}}\zeta(t,s,\boldsymbol{v},\boldsymbol{w})>\frac{2B(k,S_1,u)}{\sqrt{4-((j-1)S_1)^{\alpha}u^{-2}}}\right\},
\end{aligned}
\tag{4.38}
$$

with some $\widetilde{\triangle}_{\boldsymbol{l}}$ adjacent or identical with $\widetilde{\triangle}_{\boldsymbol{i}}$. It follows further from Theorem 3.1 that

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{\substack{(t,s)\in\triangle_0\times\triangle_j\\ \boldsymbol{v}\in\widetilde{\triangle}_{\boldsymbol{i}},\boldsymbol{w}\in\widetilde{\triangle}_{\boldsymbol{l}}}}\zeta(t,s,\boldsymbol{v},\boldsymbol{w})>\frac{2B(k,S_1,u)}{\sqrt{4-((j-1)S_1)^{\alpha}u^{-2}}}\right\}\ &\leq\ \left(\mathcal{H}_{\alpha}[0,9^{\frac{1}{\alpha}}S_1]\right)^2\left(\mathcal{H}_2[0,\sqrt{5}S_2]\right)^{2(n-1)}\\
&\quad\frac{1}{\sqrt{2\pi}u}\exp\left(-\frac{4B(k,S_1,u)^2}{2\left(4-((j-1)S_1)^{\alpha}u^{-2}\right)}\right)(1+o(1))
\end{aligned}
$$

as $u\to\infty$. Inserting the last formula into (4.38) and noting that

$$
\sharp\{\mathcal{K}_i\}=V_{n-1}(g\partial\mathcal{S}_i)S_2^{-(n-1)}u^{n-1}(1+o(1)),\quad \text{as } u\to\infty
$$

we derive that

$$
I_1(u)\ \leq\ \mathbb{Q}S_1^2S_2^{n-1}\sum_{k=0}^{h(u)-1}\sum_{j\geq 2}\frac{1}{\sqrt{2\pi}}u^{n-2}\exp\left(-\frac{u^2}{2}-c(kS_1u^{\frac{1}{\beta}-\frac{2}{\alpha}})^{\beta}-\frac{1}{8}((j-1)S_1)^{\alpha}\right).
$$

Thus, in the light of the reasoning of (3.15), we conclude that

$$
\limsup_{S_1\to\infty}\limsup_{u\to\infty}\frac{I_1(u)}{A_2(u)}\leq\mathbb{Q}\limsup_{S_1\to\infty}S_1S_2^{n-1}\exp\left(-\frac{1}{8}S_1^{\alpha}\right)=0. \tag{4.39}
$$

Moreover, in view of the reasoning of (4.36), when $\widetilde{\triangle}_{\boldsymbol{i}}\cap\widetilde{\triangle}_{\boldsymbol{l}}=\emptyset$, we obtain

$$
2\leq Var(Z(t,\boldsymbol{v},s,\boldsymbol{w}))\leq 4-((j-1)S_1)^{\alpha}u^{-2}-||\boldsymbol{l}-\boldsymbol{i}||^2S_2^2u^{-2}
$$

and thus

$$
\begin{aligned}
I_2(u)\ &\leq\ \sum_{k=0}^{h(u)-1}\sum_{j\geq 2}\sum_{\boldsymbol{i}\in\mathcal{K}_i}\sum_{\substack{\boldsymbol{l}\in\mathcal{K}_l\\ \widetilde{\triangle}_{\boldsymbol{i}}\cap\widetilde{\triangle}_{\boldsymbol{l}}=\emptyset}}\mathbb{P}\left\{\sup_{\substack{(t,s)\in\triangle_0\times\triangle_j\\ \boldsymbol{v}\in\widetilde{\triangle}_{\boldsymbol{i}},\boldsymbol{w}\in\widetilde{\triangle}_{\boldsymbol{l}}}}\zeta(t,s,\boldsymbol{v},\boldsymbol{w})>\frac{2B(k,S_1,u)}{\sqrt{4-((j-1)S_1)^{\alpha}u^{-2}-||\boldsymbol{i}-\boldsymbol{l}||^2S_2^2u^{-2}}}\right\}\\
&\leq\ \sum_{k=0}^{h(u)-1}\sum_{j\geq 2}\sum_{\boldsymbol{i}\in\mathcal{K}_i}\sum_{\substack{\boldsymbol{l}\in\mathbb{R}^{n-1}\\ \boldsymbol{l}\neq 0}}\mathbb{P}\left\{\sup_{\substack{(t,s)\in\triangle_0\times\triangle_j\\ \boldsymbol{v}\in\widetilde{\triangle}_{\boldsymbol{0}},\boldsymbol{w}\in\widetilde{\triangle}_{\boldsymbol{l}}}}\zeta(t,s,\boldsymbol{v},\boldsymbol{w})>\frac{2B(k,S_1,u)}{\sqrt{4-((j-1)S_1)^{\alpha}u^{-2}-||\boldsymbol{l}||^2S_2^2u^{-2}}}\right\}.
\end{aligned}
$$

Similar to (4.39), we conclude that

$$\limsup_{S_1\to\infty}\limsup_{u\to\infty}\frac{I_2(u)}{A_2(u)}=0,$$

hence (3.17) follows.

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References

[1] Adler, R.J. and Taylor, J.E., 2007. *Random Fields and Geometry.* Springer.

[2] Albin, J.M.P., On extremal theory for stationary processes. Ann. Probab. 18 (1990), 92-128.

[3] Albin, J.M.P., On extremal theory for self-similar processes. Ann. Probab. 26 (1998), 743-793.

[4] Albin, J.M.P. and Jarušková, D., On a test statistic for linear trend. Extremes 6 (2003), 247-258.

[5] Aronowich, M. and Adler, R. J., Behaviour of χ^2 processes at extrema. Advances in Applied Probability 17 (1985), 280-297.

[6] Belyaev, Yu. K. and Nosko, V.P., Characteristics of exursions above a high level for a Gaussian process and its envelope. Theory Probab. Appl. 13 (1969), 298-302.

[7] Berman, M.S. 1992. *Sojourns and Extremes of Stochastic Processes.* Wadsworth and Brooks/ Cole, Boston.

[8] Bojdecki, T., Gorostiza, L., and Talarczyk, A., Sub-fractional Brownian motion and its relation to occupation times. Stat. Probab. Lett. 69 (2004), 405-419.

[9] Dębicki, K., Ruin probability for Gaussian integrated processes. Stoch. Proc. Appl. 98 (2002), 151-174.

[10] Dębicki, K., Hashorva, E., and Ji, L., Tail asymptotics of supremum of certain Gaussian processes over threshold dependent random intervals. Extremes (2014), in press.

[11] Dębicki, K., Kosinski, M.K., On the infimum attained by the reflected fractional Brownian motion. Extremes (2014), in press.

[12] Dębicki, K., Sikora, G., Finite time asymptotics of fluid and ruin models: multiplexed fractional Brownian motions case. Applicationes Mathematicae 38 (2011), 107-116.

[13] Dębicki, K. and Tabiś, K., Extremes of time-average stationary Gaussian processes. Stoch. Proc. Appl. 121 (2011), 2049-2063.

[14] Dieker, A.B. and Yakir, B., On asymptotic constants in the theory of Gaussian processes. Bernoulli. to appear. (2013).

[15] Falk, M., Hüsler, J., and Reiss, R.D., 2010. *Laws of Small Numbers: Extremes and Rare Events.* DMV Seminar Vol. 23, 2nd edn., Birkhäuser, Basel.

[16] Hashorva, E., Kabluchko, Z., and Wübker, A., Extremes of independent chi-square random vectors. Extremes 15 (2012), 35–42.

[17] Houdré, C. and Villa, J., An example of infinite dimensional quasi-helix. Contemporary Mathematics, American Mathematical Society 336 (2003), 195-201.

[18] Jarušková, D., Asymptotic behaviour of a test statistic for detection of change in mean of vectors. J. Stat. Plan. Inf. 140 (2010), 616–625.

[19] Jarušková, D. and Piterbarg, V.I., Log-likelihood ratio test for detecting transient change. Stat. Probab. Lett. 81 (2011), 552-559.

[20] Kabluchko, Z., Extremes of independent Gaussian processes. Extremes 14 (2011), 285–310.

[21] Kozachenko, Y. and Moklyachuk, O., Large deviation probabilities for square-Gaussian stochastic processes. Extremes 2 (1999), 269-293.

[22] Lindgren, G., Extreme values and crossings for the χ^2-process and other functions of multidimensional Gaussian proceses with reliability applications. Adv. Appl. Probab. 12 (1980a), 746-774.

[23] Lindgren, G., Point processes of exits by bivariate Gaussian processes and extremal theory for the χ^2-process and its concominats. J. Multivar. Anal. 10 (1980b), 181-206.

[24] Lindgren, G., Slepian models for χ^2-process with dependent components with application to envelope upcrossings. J. Appl. Probab. 26 (1989), 36-49.

[25] Michna, Z., Remarks on Pickands theorem. Preprint. (2009). Available at http://arxiv.org/abs/0904.3832

[26] Pickands III, J., Upcrossing probabilities for stationary Gaussian processes. Trans. Amer. Math. Soc. 145 (1969a), 51-73.

[27] Pickands III, J., Asymptotic properties of the maximum in a stationary Gaussian process. Trans. Amer. Math. Soc. 145 (1969b), 75-86.

[28] Piterbarg, V.I., On the paper by J. Pickands "Upcrosssing probabilities for stationary Gaussian processes", Vestnik Moscow. Univ. Ser. I Mat. Mekh. 27 (1972), 25-30. English transl. in Moscow Univ. Math. Bull., 27 (1972).

[29] Piterbarg, V.I., High deviations for multidimensional stationary Gaussian processes with independent components. In: Zolotarev,V.M.(Ed.), Stability Problems for Stochastic Models (1994a), 197-210.

[30] Piterbarg, V.I., High excursions for nonstationary generalized chi-square processes. Stoch. Proc. Appl. 53 (1994b), 307-337.

[31] Piterbarg, V.I., 1996. *Asymptotic Methods in the Theory of Gaussian Processes and Fields.* In: Transl. Math. Monographs, vol. 148. AMS, Providence, RI.

[32] Piterbarg, V.I., Large deviations of a storage process with fractional Browanian motion as input. Extremes 4 (2001), 147-164.

[33] Piterbarg, V.I., Prisyazhnyuk, V., Asymptotic behavior of the probability of a large excursion for a nonstationary Gaussian processes. Teor. Veroyatnost. i Mat. Statist. 18 (1978), 121-133.